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Controlling Chaos in Two Lorenz Systems by Delayed Coupling

Jin-Song Kang, Jong-Myong Ri, Ryong-Son Kim, and Chol-Ung Choe_
Research Group for Nonlinear Dynamics, Department of Physics, University of Science,
Unjong-District, Pyongyang, Democratic People's Republic of Korea

We study two delay-coupled Lorenz systems and demonstrate unified chaos control by noninvasive timedelayed coupling. Both an unstable periodic orbit and an unstable fixed point of the system can be stabilized close to a subcritical Hopf bifurcation. Using a multiple scales method, the systems are reduced to Hopf normal forms, and an analytical approach for stabilizing a periodic orbit as well as a fixed point of the system is developed. As a result, the equations for the characteristic exponents are derived in an analytical form, revealing the range of coupling parameters for successful stabilization. Finally, we illustrate the results with numerical simulations, which show good agreement with the theory.

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I. INTRODUCTION

Time delayed feedback control (DFC), proposed by Pyragas [1], is a simple and convenient method to stabilize unstable periodic orbits (UPOs) occurring in a single dynamical system. Since the DFC uses only the difference of the current and the delayed state where the time delay is given by the period of the UPO, the control is non-invasive and is applicable to systems whose equations of motion are unknown. Due to this convenience, the algorithm of DFC has been applied to quite diverse experimental systems and theoretical advances have also been made [2–6].

However, it was commonly believed that torsion-free UPOs or, more precisely, UPOs with an odd-number of real Floquet multipliers larger than unity could not be stabilized by DFC [7, 8]. To overcome this limitation, the modified control schemes, like a half-period delay [9] and the introduction of a unstable controller [10, 11] were proposed for stabilizing UPOs of the Lorenz system that is a representative with the odd number limitation.

This alleged odd-number theorem has been refuted by a counter example, i.e., subcritical Hopf-normal form system [12], and this refuting mechanism has been applied directly to UPOs created by subcritical Hopf bifurcations in Lorenz system, higher-dimensional dynamical systems [13]. When the delay term is added, however, reducing Lorenz system to the standard normal form via the center manifold theory is nontrivial task because the dynamics takes place in an infinite-dimensional phase space [11]. In [14], we have shown that UPOs in the network of the subcritical Hopf bifurcation systems can be stabilized in-phase synchronously by delayed coupling, which has been extended to network of Lorenz systems by reduction using the method of multiple scale [15].

In parallel to the control of UPOs, the stabilization of unstable steady states (USSs) has become a field of increasing interest. Although the field of controlling chaos deals mainly with the stabilization of UPOs, the problem of controlling the system dynamics on USSs could be of practical importance in experimental situations where chaotic or periodic oscillations cause degradation in performance. One of the methods to control an USS introduced by Pyragas *et al.* uses the difference between the current state and a low-pass filtered version, in which an unstable degree of freedom in the feedback loop of Lorenz system was added to overcome the topological limitation, similar to that of a time-delay feedback controller [16, 17]. A DFC scheme in a diagonal coupling form, which was originally invented to control UPOs, has been analytically investigated for the purpose of stabilizing the USS [18, 19]. For stronger couplings a diffusively coupled limit cycle oscillators with time delay exhibit a coupling induced stabilization of an USS, that is, amplitude death of the oscillations [20, 21].

In our previous study [22], we proposed a time-delayed coupling method which makes it possible to stabilize not only UPO but also USS in two delay-coupled Hopf normal form systems as a result of conversion of stability. In this paper, we extend this idea into Lorenz system as a representative of troublesome dynamical systems from the viewpoint of controlling chaos as described above. We consider two delay-coupled Lorenz systems and develop a systematic analytical approach for delayed coupling control of dynamical systems close to a subcritical Hopf bifurcation. Using the multiple scale method, the system is reduced to the normal form for Hopf bifurcation and equations for the characteristic exponents are derived in an analytical form, which reveal the coupling parameters for successful stabilization. As a result of this we can show that UPO and USS in Lorenz system can be stabilized by delay-coupling. Finally, we illustrate the results with numerical simulations of Lorenz system close to a subcritical Hopf bifurcation.

The paper is organized as follows. In Sec. II we present our model and an outline of the stability diagram. Section III is devoted to reducing our model into a delay-coupled Hopf normal form systems using the method of multiple scale. In Sec. IV, we derive analytical stability conditions and confirm its validity with the direct numerical simulations. Finally, Sec. V, concludes the paper.

II. A MODEL OF TWO DELAY-COUPLED LORENZ SYSTEMS

We consider the following model of two delay-coupled Lorenz systems:

$$\dot{\mathbf{x}}_1 = \mathbf{F}(\mathbf{x}_1;\rho_1) + kH(\mathbf{x}_2 - \mathbf{x}_2^*), \tag{1a}$$

$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_2;\rho_2) - kH(\mathbf{x}_1 - \mathbf{x}_{1,\tau}), \tag{1b}$$

where

$$\mathbf{F}(\mathbf{x};\rho) \equiv \begin{pmatrix} \sigma y - \sigma x \\ \rho x - y - xz \\ xy - \mathrm{b}z \end{pmatrix}$$

describes the Lorenz system with state vector $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and the real parameters σ, ρ and b. We select the standard set of the parameter values, i.e., $\sigma = 10$, $\mathrm{b} = 8/3$, and the bifurcation parameter ρ is assumed to vary. k denotes the coupling strength, the 3×3 matrix H is the connectivity matrix that determines which components of the vector $\mathbf{x}_j$ enter the coupling, and $\mathbf{x}_{1,\tau} = \mathbf{x}_1(t-\tau)$ with time delay τ.

It is well known [23, 24] that the original Lorenz equations with $k = 0$ demonstrate different dynamical regimes on variation of the control parameter ρ, which are associated with the existence and stability of several equilibrium states. In brief, the system dynamics can be characterized by three regimes. For $0 < \rho < 1$, there exists the only stable fixed point at the origin $\mathbf{x} = (0, 0, 0)$. For $\rho > 1$, the origin becomes a saddle and two additional symmetrical stable fixed points $\mathbf{x} = \mathbf{x}_\pm^*(\rho) \equiv (\pm\sqrt{\mathrm{b}(\rho-1)}, \pm\sqrt{\mathrm{b}(\rho-1)}, \rho-1)$ appear. For $\rho > \rho_H \approx 24.7368$, the steady states become unstable through the subcritical Hopf bifurcation at $\rho = \rho_H$. Just below this bifurcation point, for $\rho = \rho_H + \epsilon$, $\epsilon < 0$, there are two small unstable limit cycles $\tilde{\mathbf{x}}_\pm(t)$ surrounding the stable steady states $\mathbf{x}_\pm^*$. At the same values of the parameter ρ there exists a strange attractor and thus the system is multistable depending on initial conditions, i.e., the phase trajectory may either be attracted to the one of the steady states or exhibit a chaotic behavior on the strange attractor. The periodic orbit exists for $13.926 < \rho < \rho_H$; at the lower boundary it collides with a fixed point in a homoclinic bifurcation.

In the following, the value of the bifurcation parameter for the $\mathbf{x}_2$-system which plays the role of controller is taken by $\rho_2 > \rho_H$, and thus its fixed points become USSs. Whereas, the $\mathbf{x}_1$-system to be controlled might exhibit UPO and USS according to the choice of parameter values of $\rho_1 < \rho_H$ and $\rho_1 > \rho_H$, respectively. Our aim is to stabilize a UPO/USS of $\mathbf{x}_1$-system and a USS of $\mathbf{x}_2$-system all together in two delay-coupled Lorenz systems (1) by choosing the proper delay-time and coupling strength.

III. REDUCING SYSTEMS TO NORMAL FORM

In this section, we transform the variables of the coupled system using the eigenvectors of the fixed point at the bifurcation point as a basis for a new coordinate system. Then applying the method of multiple scales we eliminate the decaying mode and obtain the normal form for oscillating modes with delayed-coupling.

A. Transforming the System Variables

First the origins of both phase space and bifurcation parameter are shifted to a fixed point, e.g., $\mathbf{x}_j^*(\rho) = \mathbf{x}_{j+}^*$ and Hopf bifurcation point ρ_H by using the transformations $\mathbf{x}_j(t) = \mathbf{x}_j^* + \mathbf{u}_j(t)$ and $\rho_j = \rho_H + \epsilon_j$, respectively, with $j = 1, 2$. We rewrite Eq. (1) in the form

$$\dot{\mathbf{u}}_1 = A^{(0)}\mathbf{u}_1 + \epsilon_1 A^{(1)}\mathbf{u}_1 + N(\mathbf{u}_1) + kH\mathbf{u}_2, \tag{2a}$$

$$\dot{\mathbf{u}}_2 = A^{(0)}\mathbf{u}_2 + \epsilon_2 A^{(1)}\mathbf{u}_2 + N(\mathbf{u}_2) - kH(\mathbf{u}_1 - \mathbf{u}_{1,\tau}), \tag{2b}$$

where $A^{(0)}$ is the Jacobian evaluated at the point $(\mathbf{x}_j^*(\rho_H))$, $\epsilon_j A^{(1)}$ is a small deviation due to the shift of the parameter ρ_j from the bifurcation point ρ_H, and $N(\mathbf{u}_j)$ defines the nonlinear part. The Lorenz system yields the following matrices:

$$A^{(0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -p \\ p & p & -\mathrm{b} \end{pmatrix},\ A^{(1)} = q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix},\ N(\mathbf{u}_j) = \begin{pmatrix} 0 \\ -\mathrm{u}_{j1}\mathrm{u}_{j3} \\ \mathrm{u}_{j1}\mathrm{u}_{j2} \end{pmatrix}, \tag{3}$$

where $p = \sqrt{\mathrm{b}(\rho_H-1)}$, $q = \sqrt{\mathrm{b}/(\rho_H-1)}/2$ and the approximation $\sqrt{\mathrm{b}(\rho-1)} - \sqrt{\mathrm{b}(\rho_H-1)} \approx \epsilon\sqrt{\mathrm{b}/(\rho_H-1)}/2$ was used.

Let Φ be the matrix that transforms the matrix $A^{(0)}$ into Jordan canonical form, i.e., the columns of the matrix Φ are the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ of the matrix $A^{(0)}$ corresponding to the eigenvalues γ_1, γ_2, γ_3, respectively. Solving the eigenvalue problem for the matrix $A^{(0)}$, we obtain three eigenvalues

$$\gamma_1 = \gamma_2^* \equiv i\omega_0 \approx 9.624i,\ \ \gamma_3 \approx -13.666 \tag{4}$$

and the matrix $\Phi = [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]$ reads

$$\Phi = \begin{pmatrix} 0.268 + 0.306i & 0.268 - 0.306i & 0.863 \\ -0.027 + 0.564i & -0.027 - 0.564i & -0.316 \\ 0.7187 & 0.7187 & -0.395 \end{pmatrix}. \tag{5}$$

Then, under the transformation $\mathbf{u}_j(t) = \Phi\mathbf{v}_j(t) = \sum_{m=1}^{3} \mathrm{v}_{j,m}(t)\mathbf{p}_m$ Eq. (2) yields the eigenmode equations as follows

$$\dot{\mathbf{v}}_1 = J\mathbf{v}_1 + \epsilon_1 A\mathbf{v}_1 + B\mathbf{v}_1^2 + \epsilon_1 k_1 C\mathbf{v}_2, \tag{6a}$$

$$\dot{\mathbf{v}}_2 = J\mathbf{v}_2 + \epsilon_2 A\mathbf{v}_2 + B\mathbf{v}_2^2 - \epsilon_2 k_2 C(\mathbf{v}_1 - \mathbf{v}_{1,\tau}), \tag{6b}$$

where $J = \Phi^{-1}A^{(0)}\Phi = \mathrm{diag}\,(\gamma_1, \gamma_2, \gamma_3)$, $A = \Phi^{-1}A^{(1)}\Phi$, and $C = \Phi^{-1}H\Phi$ are the 3×3 similarity matrices by Φ of $A^{(0)}$, $A^{(1)}$ and H, respectively. The notation $B\mathbf{v}_j^2 \equiv \Phi^{-1}N(\Phi\mathbf{v}_j)$ can be regarded by product of the 3×6 matrix B and the column vector with $6(=\mathrm{P}_3^2)$ elements, $\mathbf{v}_j^2 \equiv (\mathrm{v}_{j1}^2,\ \mathrm{v}_{j1}\mathrm{v}_{j2},\ \mathrm{v}_{j2}^2,\ \mathrm{v}_{j1}\mathrm{v}_{j3},\ \mathrm{v}_{j2}\mathrm{v}_{j3},\ \mathrm{v}_{j3}^2)^T$. The coupling strength in Eqs. (6) was rescaled as $k = \epsilon_j k_j$ in order that the influences of coupling terms are realized at the same order with the bifurcation parameter term $\epsilon_j A\mathbf{v}_j$.

Since a pair of eigenvalues are complex conjugate, i.e., $\gamma_2 = \gamma_1^*$, the corresponding eigenvectors are also complex conjugate: $\mathbf{p}_2 = \mathbf{p}_1^*$. Moreover, the amplitudes of the corresponding eigenmodes have to be complex conjugate, $\mathrm{v}_{j2} = \mathrm{v}_{j1}^*$, in order to provide the *real* valued solution for $\mathbf{u}_j(t)$. Therefore, it is enough to observe only one of the two eigenmodes, e.g., v_{j1}.

In the following numerical simulations, we will use z-coupling with the connectivity matrix,

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

According to the expressions (3) and (7), the matrices A, B and C are given as follow as

$$A = \begin{pmatrix} 0.03 + 0.18i & 0.049 + 0.005i & 0.047 - 0.044i \\ 0.049 - 0.005i & 0.03 - 0.18i & 0.047 + 0.044i \\ 0.042 - 0.048i & 0.042 + 0.048i & -0.06 \end{pmatrix}, \tag{8a}$$

$$B = \begin{pmatrix} -0.26 + 0.25i & 0.25 + 0.31i & 0.09 + 0.02i & 0.07 + 0.6i & -0.07 + 0.16i & -0.21 - 0.27i \\ 0.09 - 0.02i & 0.25 - 0.31i & -0.26 - 0.25i & -0.07 - 0.16i & 0.07 - 0.6i & -0.21 + 0.27i \\ 0.15 + 0.05i & 0.07 & 0.15 - 0.05i & 0.27 - 0.18i & 0.27 + 0.18i & -0.07 \end{pmatrix}, \tag{8b}$$

$$C = \begin{pmatrix} 0.4348 + 0.0459i & 0.4348 + 0.0459i & -0.239 - 0.0252i \\ 0.4348 - 0.0459i & 0.4348 - 0.0459i & -0.239 + 0.0252i \\ -0.2373 & -0.2373 & 0.1304 \end{pmatrix}. \tag{8c}$$

B. Application of the multiple scales method

We now begin the task of simplifying Eq. (6), i.e., reducing the dimensionality and eliminating the nonlinearity in the term $B\mathbf{v}_j^2$ as much as possible. For doing that, we apply an approximation, the method of multiple scales [25], seeking an expansion of the form

$$\mathrm{v}_{j1} = \sum_{l=1}^{3} \mu^l \xi_{jl}(T_0, T_1, T_2) + O(\mu^4), \tag{9a}$$

$$\mathrm{v}_{j3} = \sum_{l=1}^{3} \mu^l \eta_{jl}(T_0, T_1, T_2) + O(\mu^4), \tag{9b}$$

where the time scales T_l are defined by $T_l = \mu^l t$ and μ is a small positive dimensionless parameter that is artificially introduced to establish the different orders of magnitude. The results obtained are independent of this parameter, and it is ultimately absorbed back into the solution, which is equivalent to setting it equal to unity in the final analysis. In terms of the T_l, the time derivative becomes

$$\frac{d}{dt} = D_0 + \mu D_1 + \mu^2 D_2 + \cdots \tag{10}$$

where $D_l = \partial/\partial T_l$. Also, the parameter ϵ is ordered as $\epsilon_j = \mu^2\epsilon_j'$, so that the influences of the nonlinear terms, coupling terms and the bifurcation parameter term $\epsilon_j A\mathbf{v}_j$ are realized at the same order.

Substituting Eqs. (9) and (10) into Eqs. (6), and equating coefficients of like powers of μ, we obtain the hierarchy of equations.

$O(\mu)$:

$$D_0\xi_{j1} - i\omega_0\xi_{j1} = 0, \tag{11a}$$

$$D_0\eta_{j1} - \gamma_3\eta_{j1} = 0 \tag{11b}$$

The nondecaying solution of Eqs. (11) is

$$\xi_{j1} = W_j(T_1, T_2)e^{i\omega_0 T_0}, \tag{12a}$$
$$\eta_{j1} = 0, \tag{12b}$$

where W_j is determined by imposing the solvability conditions at the next levels of approximation.

$O(\mu^2)$:

$$D_0\xi_{j2} - i\omega_0\xi_{j2} = b_{11}\xi_{j1}^2 + b_{12}\xi_{j1}\xi_{j1}^* + b_{13}\xi_{j1}^{*2} - D_1\xi_{j1}, \tag{13a}$$
$$D_0\eta_{j2} - \gamma_3\eta_{j2} = b_{31}\xi_{j1}^2 + b_{32}\xi_{j1}\xi_{j1}^* + b_{33}\xi_{j1}^{*2} \tag{13b}$$

Substituting Eqs. (12) into Eqs. (13) and eliminating the source of secular terms, we have $D_1W_j = 0$ or $W_j = W_j(T_2)$. Then, the solutions of Eqs. (13) are

$$\xi_{j2} = \frac{b_{11}W_j^2 e^{2i\omega_0 T_0}}{i\omega_0} - \frac{b_{12}W_jW_j^*}{i\omega_0} - \frac{b_{13}W_j^{*2}e^{-2i\omega_0 T_0}}{3i\omega_0}, \tag{14a}$$
$$\eta_{j2} = -\left(\frac{b_{31}W_j^2 e^{2i\omega_0 T_0}}{\gamma_3 - 2i\omega_0} + \frac{b_{32}W_jW_j^*}{\gamma_3} + \frac{b_{33}W_j^{*2}e^{-2i\omega_0 T_0}}{\gamma_3 + 2i\omega_0}\right), \tag{14b}$$

where the general solutions of the homogeneous equations of (13) were omitted since they have no influence on the source of secular terms in the next level.

$O(\mu^3)$:

$$D_0\xi_{13} - i\omega_0\xi_{13} = \epsilon_1'(a_{11}\xi_{11} + a_{12}\xi_{11}^*) + 2b_{11}\xi_{11}\xi_{12} + b_{12}(\xi_{11}\xi_{12}^* + \xi_{11}^*\xi_{12}) + 2b_{13}\xi_{11}^*\xi_{12}^* + b_{14}\xi_{11}\eta_{12} + b_{15}\xi_{11}^*\eta_{12} - D_2\xi_{11} + \epsilon_1' k_1(c_{11}\xi_{21} + c_{12}\xi_{21}^*), \tag{15a}$$
$$D_0\xi_{23} - i\omega_0\xi_{23} = \epsilon_2'(a_{11}\xi_{21} + a_{12}\xi_{21}^*) + 2b_{11}\xi_{21}\xi_{22} + b_{12}(\xi_{21}\xi_{22}^* + \xi_{21}^*\xi_{22}) + 2b_{13}\xi_{21}^*\xi_{22}^* + b_{14}\xi_{21}\eta_{22} + b_{15}\xi_{21}^*\eta_{22} - D_2\xi_{21} + \epsilon_2' k_2\left[c_{11}(\xi_{11,\tau} - \xi_{11}) + c_{12}(\xi_{11,\tau}^* - \xi_{11}^*)\right], \tag{15b}$$

where $\xi_{11,\tau} = \xi_{11}(T_0 - \tau,\ T_1 - \mu\tau,\ T_2 - \mu^2\tau)$.

Substituting Eqs. (14) into Eq. (15) and eliminating the terms that produce secular terms, we obtain equations for the slowly varying amplitude

$$\frac{dW_1}{dT_2} = (\epsilon_1' a + b\,|W_1|^2)W_1 + \epsilon_1' k_1 c W_2, \tag{16a}$$
$$\frac{dW_2}{dT_2} = (\epsilon_2' a + b\,|W_2|^2)W_2 + \epsilon_2' k_2 c(\tilde{W}_{1,\mu^2\tau} - W_1), \tag{16b}$$

where $\tilde{W}_{1,\mu^2\tau} = e^{-i\omega_0\tau}W_1(T_2 - \mu^2\tau)$ and the complex parameters are given as follow:

$$a = a_{11}, \tag{17a}$$
$$b = \frac{i}{\omega_0}(b_{11}b_{12} - b_{12}b_{12}^* - \frac{2}{3}b_{13}b_{13}^*) - \frac{b_{14}b_{32}}{\gamma_3} - \frac{b_{15}b_{31}}{\gamma_3 - 2i\omega_0}, \tag{17b}$$
$$c = c_{11}. \tag{17c}$$

Multiplying Eq. (16) with $e^{i\omega_0 T_0}$ and setting $\mu = 1$, we finally arrive at two delay-coupled Hopf bifurcation systems as follow as

$$\dot{Z}_1 = \left(\lambda_1 + i\omega_1 + (b_R + ib_I)\,|Z_1|^2\right)Z_1 + Ke^{i\beta}Z_2, \tag{18a}$$
$$\dot{Z}_2 = \left(\lambda_2 + i\omega_2 + (b_R + ib_I)\,|Z_2|^2\right)Z_2 - Ke^{i\beta}\left(Z_1 - Z_{1,\tau}\right) \tag{18b}$$

where $Z_j(t) = e^{i\omega_0 t}W_j(t)$, $\lambda_j = \epsilon_j a_R$, $\omega_j = \omega_0 + \epsilon_j a_I$, $a_R = \mathrm{Re}(a)$, $a_I = \mathrm{Im}(a)$, $b_R = \mathrm{Re}(b)$, $b_I = \mathrm{Im}(b)$, $\beta = \arg(c)$ and $K = k|c|$.

The parameters a, b, c and ω_0 determine, in whole, the coupled system (18) of Hopf-normal forms. According to Eqs. (8) and (17), the values of the parameters are given as follow as

$$a_R = 0.03022, \tag{19a}$$
$$a_I = 0.18145, \tag{19b}$$
$$b_R = 0.00256, \tag{19c}$$
$$b_I = -0.02765, \tag{19d}$$
$$|c| = 0.4372, \tag{19e}$$
$$\beta = 0.105244, \tag{19f}$$
$$\omega_0 = 9.624, \tag{19g}$$

which are used for the numerical simulations in the following section.

For $b_R > 0$, Eqs. (18) without coupling ($K = 0$) describe the normal forms for subcritical Hopf bifurcation with bifurcation parameter λ_j, i.e., ϵ_j. UPOs with radius $r_j^* = \sqrt{-\lambda_j/b_R}$ and period $T_j = 2\pi/\Omega_j = 2\pi/(\omega_j - \lambda_j b_I/b_R)$ exist for $\lambda_j < 0$. The FEs of the UPO are determined by $\Lambda_j^0 = -2\lambda_j$. For $\lambda_j > 0$, clearly, there is no limit cycle, and the origin $Z_j = 0$ is USS with the characteristic exponents $\lambda_j + i\omega_j$.

IV. LINEAR STABILITY ANALYSIS

In this section, we analyze Eqs. (18) and demonstrate stabilization of both an UPO and an USS by using the delayed coupling. In the previous study [22], we analyzed Eqs. (18) approximately in case that both the phase shift b_I and the coupling phase β are zero, of which the approach is no longer available for the present non-zero case.

The reduced system (18) in the Hopf normal form admits an analytical treatment to derive the equation for the FEs and the stability conditions. In addition, the numerical analysis of the original system of nonlinear differential-difference Eqs. (1) is performed to confirm the analytical results. The bifurcation parameter of $\mathbf{x}_2$-system is fixed by $\rho_2 > \rho_H$, i.e., $\epsilon_2 > 0$, while the $\mathbf{x}_1$-system takes the parameter value either at $\rho_1 < \rho_H$ or $\rho_1 > \rho_H$ when stabilizing a UPO or a USS is considered, respectively.

A. Stabilization of UPO and USS for $\rho_1 < \rho_H$ and $\rho_2 > \rho_H$

First, consider stabilization of UPO in the *reduced* system Eqs. (18) with $\lambda_1 < 0$ and $\lambda_2 > 0$. The time delay τ is chosen to be equal to the period of UPO, which allows for noninvasive control of the dynamical systems.

Calculating the Floquet exponents (FEs) Λ of UPO is not straightforward since Eqs. (18a) and (18b) should be linearized around a UPO and a USS, respectively. Introducing the real amplitude r_1 and phase φ_1 by $Z_1(t) = r_1(t)e^{i\varphi_1(t)}$, we obtain the following equations:

$$\dot{r}_1 = (\lambda_1 + b_R r_1^2)r_1 + K\mathrm{Re}(e^{i(\beta-\varphi_1)}Z_2), \tag{20a}$$

$$\dot{\varphi}_1 = \omega_1 + b_I r_1^2 + \frac{K}{r_1}\mathrm{Im}(e^{i(\beta-\varphi_1)}Z_2), \tag{20b}$$

$$\dot{Z}_2 = (\lambda_2 + i\omega_2)Z_2 + Ke^{i\beta}(r_1 e^{i\varphi_1} - r_{1,\tau}e^{i\varphi_{1,\tau}}), \tag{20c}$$

where the cubic term of Z_2 was neglected since we confine ourselves to the behavior close to the USS.

Using the ansatz $r_1(t) = r_1^*(1+\delta r_1(t))$, $\varphi_1(t) = \Omega_1 t + \delta\varphi_1(t)$ and $Z_2 = 0 + r_1^*\delta z_2$, expanding Eqs. (20) to linear order in the small deviations δr_1, $\delta\varphi_1$ and δz_2 around the periodic orbit, we obtain

$$\delta\dot{r}_1 = \Lambda_1^0\delta r_1 + K\mathrm{Re}(e^{i(\beta-\Omega_1 t)}\delta z_2), \tag{21a}$$

$$\delta\dot{\varphi}_1 = 2b_I r_1^*\delta r_1 + K\mathrm{Im}(e^{i(\beta-\Omega_1 t)}\delta z_2), \tag{21b}$$

$$\delta\dot{z}_2 = (\lambda_2 + i\omega_2)\delta z_2 + Ke^{i(\beta+\Omega_1 t)}\left[\delta r_{1,\tau} - \delta r_1 + i(\delta\varphi_{1,\tau} - \varphi_1)\right], \tag{21c}$$

where $\Lambda_1^0 = \lambda_1 + 3b_R r_1^{*2} = -2\lambda_1$ is the FE for UPO of the decoupled free system, and the expression $\delta z_1 = r_1^*(1+\delta r_1)e^{i(\Omega_1 t+\delta\varphi_1)} - r_1^* e^{i\Omega_1 t} = r_1^* e^{i\Omega_1 t}(\delta r_1 + i\delta\varphi_1)$ was used.

Using transformation of $Z_2(t)$ to corotating complex coordinates $\zeta(t) = e^{i(\beta-\Omega_1 t)}Z_2$, Eq. (21c) reads

$$\delta\dot{\zeta} = (\lambda_2 + i\Delta\bar{\omega})\delta\zeta + Ke^{2i\beta}\left[\delta r_{1,\tau} - \delta r_1 + i(\delta\varphi_{1,\tau} - \delta\varphi_1)\right],$$

which can be rewritten in terms of real valued coordinates $\delta\zeta = \delta\zeta_R + i\delta\zeta_I$ as follows as:

$$\begin{pmatrix} \delta\dot{\zeta}_R \\ \delta\dot{\zeta}_I \end{pmatrix} = \begin{pmatrix} \lambda_2 & -\Delta\bar{\omega} \\ \Delta\bar{\omega} & \lambda_2 \end{pmatrix}\begin{pmatrix} \delta\zeta_R \\ \delta\zeta_I \end{pmatrix} + K\begin{pmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{pmatrix}\begin{pmatrix} \delta r_{1,\tau} - \delta r_1 \\ \delta\varphi_{1,\tau} - \delta\varphi_1 \end{pmatrix}, \tag{22}$$

where $\Delta\bar{\omega} = \omega_2 - \Omega = \Delta\omega - b_I r^{*2}$ with $\Delta\omega = \omega_2 - \omega_1$.

Since the coefficient matrices of Eqs. (21a), (21b) and (22) do not depend on time, the FEs of the periodic orbit are simply given by the eigenvalues Λ of the characteristic equation

$$\begin{vmatrix} \Lambda_1^0 - \Lambda & 0 & K & 0 \\ 2b_I r_1^{*2} & -\Lambda & 0 & K \\ K\chi_c(\Lambda) & -K\chi_s(\Lambda) & \lambda_2 - \Lambda & -\Delta\bar{\omega} \\ K\chi_s(\Lambda) & K\chi_c(\Lambda) & \Delta\bar{\omega} & \lambda_2 - \Lambda \end{vmatrix} = 0, \tag{23}$$

where $\chi_c(\Lambda) = (e^{\Lambda\tau} - 1)\cos(2\beta)$, $\chi_s(\Lambda) = (e^{\Lambda\tau} - 1)\sin(2\beta)$. Clearly, the characteristic equation admits the solution $\Lambda = 0$, which corresponds to the trivial Floquet mode of UPO with Floquet multiplier 1. In the absence of the coupling, $K = 0$, Eq. (23) yields the quartic equation

$$\Lambda(\Lambda - \Lambda_1^0)\left[(\Lambda - \lambda_2)^2 + \Delta\bar{\omega}^2\right] = 0.$$

If we assume the diagonal coupling, $\beta = 0$, and $\Delta\bar{\omega} \approx 0$, then the characteristic equation (23) factorizes:

$$\left[\Lambda(\Lambda - \lambda_2) + K^2(1 - e^{-\Lambda\tau})\right]\left[(\Lambda - \Lambda_1^0)(\Lambda - \lambda_2) + K^2(1 - e^{-\Lambda\tau})\right] = 0. \tag{24}$$

In order to appreciate the behavior of the roots at a rough estimate, we briefly reduce Eq. (24) to a polynomial equation using an approximation $e^{-\Lambda\tau} \approx 1 - \Lambda\tau$ for small $|\Lambda|\tau$ as follows as

$$\Lambda(\Lambda + \kappa - \lambda_2)\left[\Lambda^2 + (\kappa - \Lambda_1^0 - \lambda_2)\Lambda + \Lambda_1^0\lambda_2\right] = 0, \tag{25}$$

where $\kappa = K^2\tau$. The roots of the second and third factors give the nontrivial FEs, of which the former crosses into the left half plane at $\kappa = \kappa_1 \equiv \lambda_2$ and the latters show the loci as considered in [16]: For $\kappa = 0$, the FEs are Λ_1^0 and λ_2. With the increase of κ, they approach each other on the real axis, then collide at $\kappa = \Lambda_1^0 + \lambda_2 - 2\sqrt{\Lambda_1^0\lambda_2}$ and form a complex conjugate pair in the complex plane. At $\kappa = \kappa_2 \equiv \Lambda_1^0 + \lambda_2$, they cross symmetrically into the left half plane (inverse Hopf bifurcation). Taking into account $\kappa_2 > \kappa_1$ due to $\Lambda_1^0 = -2\lambda_1 > 0$, the dominant FEs are determined by quadratic equation,

$$\Lambda^2 + (\kappa - \Lambda_1^0 - \lambda_2)\Lambda + \Lambda_1^0\lambda_2 = 0, \tag{26}$$

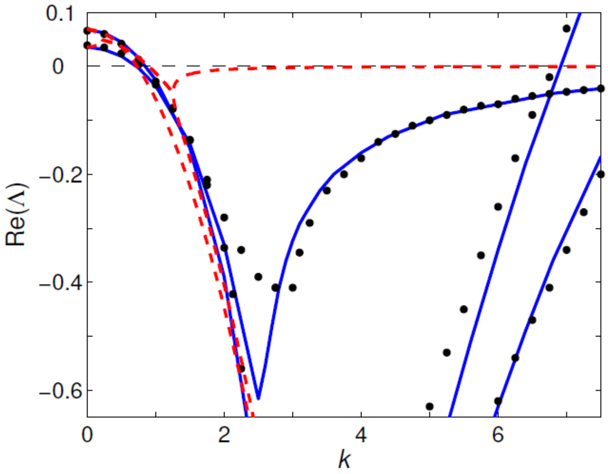

FIG. 1: (Color online) Real parts of leading Floquet exponents Λ of UPO in delay-coupled Lorenz systems as a function of coupling strength k. The dashed (red) and solid (blue) lines denote the solutions of the polynomial Eq. (25) and the transcendental Eq. (23), respectively. Dots correspond to Re Λ obtained from the exact variational Eqs. (27). Parameters are given by $\rho_1 = 24.144$ (i.e., $\lambda_1 = -0.01792$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.0684$), $\tau = 0.674$, and Eqs. (7) and (19).

which provides the mechanism of stabilization of UPO and yields the stability condition $K^2\tau > -2\lambda_1 + \lambda_2$.

However, such qualitative estimations using the polynomial approximation should be confirmed through the solutions of the transcendental equation Eq. (23) and, more correctly, through the numerical calculation of the variational equations of the exact systems Eqs. (1).

Next, we determine the *exact* FEs by linearization of Eqs. (1) around UPO for $\rho_1 < \rho_H$ and $\rho_2 > \rho_H$:

$$\delta\dot{\mathbf{x}}_1 = \tilde{A}_1(t)\delta\mathbf{x}_1 + kH\delta\mathbf{x}_2, \tag{27a}$$

$$\delta\dot{\mathbf{x}}_2 = \bar{A}_2\delta\mathbf{x}_2 - kH\left(\delta\mathbf{x}_1 - \delta\mathbf{x}_{1,\tau}\right), \tag{27b}$$

where $\tilde{A}_1(t) = D_{\mathbf{x}}\mathbf{F}(\tilde{\mathbf{x}}_1(t);\rho_1)$ and $\bar{A}_2 = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_2^*;\rho_2)$. $D_{\mathbf{x}}\mathbf{F}$ denotes the matrix of first partial derivatives of $\mathbf{F}$ with respect to the vector arguments. Here $\tilde{A}_1(t)$ that is taken on UPO, $[\tilde{x}_1(t), \tilde{y}_1(t), \tilde{z}_1(t)] = [\tilde{x}_1(t+\tau), \tilde{y}_1(t+\tau), \tilde{z}_1(t+\tau)]$, is τ-period 3×3 matrix and $\delta\mathbf{x} = (\delta\mathbf{x}_1, \delta\mathbf{x}_2)$ denotes small deviation from the periodic orbit $\tilde{\mathbf{x}}(t) = (\tilde{\mathbf{x}}_1(t), \mathbf{x}_2^*)$ which satisfies the decoupled system.

The FEs of the exact variational Eq. (27) have been calculated by the algorithm described in [26] as follow as. According to the Floquet theory, solutions of Eq. (27) can be decomposed into eigenfunctions

$$\delta\mathbf{x} = e^{\Lambda t}\mathbf{w}(t), \quad \mathbf{w}(t) = \mathbf{w}(t+\tau),$$

and the delay term can be eliminated, $\delta\mathbf{x}(t-\tau) = e^{-\Lambda\tau}\delta\mathbf{x}(t)$. The characteristic equation for the FEs reads

$$\det\left\{\Psi(\Lambda,\tau) - \exp\left(\Lambda\tau\right)I\right\} = 0, \tag{28}$$

where I is the 6×6 identity and matrix $\Psi(\Lambda, t)$ is the fundamental matrix of Eq. (27) that is defined by the equalities

$$\dot{\Psi}(\Lambda,t) = \left[\tilde{A}(t) + G(\Lambda)\right]\Psi(\Lambda,t), \quad \Psi(\Lambda,0) = I$$

with $\tilde{A}(t) = \begin{pmatrix} \tilde{A}_1(t) & \mathbf{0} \\ \mathbf{0} & \bar{A}_2 \end{pmatrix}$ and $G(\Lambda) = k\begin{pmatrix} 0 & 1 \\ (e^{-\Lambda\tau}-1) & 0 \end{pmatrix} \otimes H$. Here $\mathbf{0}$ is the 3×3 null matrix and $\otimes$ is the direct product.

In Figure 1, we compare the real parts of the FEs Λ as a function of the coupling gain k, which were determined by three different methods, namely, (i) using the solutions of the quadratic equation (25) in dashed (red) line, (ii) by solving the transcendental equation (23) in solid (blue) line, and (iii) by solving Eq. (28) for the exact FEs of the system (1) in the black dots. Indeed, we see that there exists an interval of coupling gain k for which the real parts

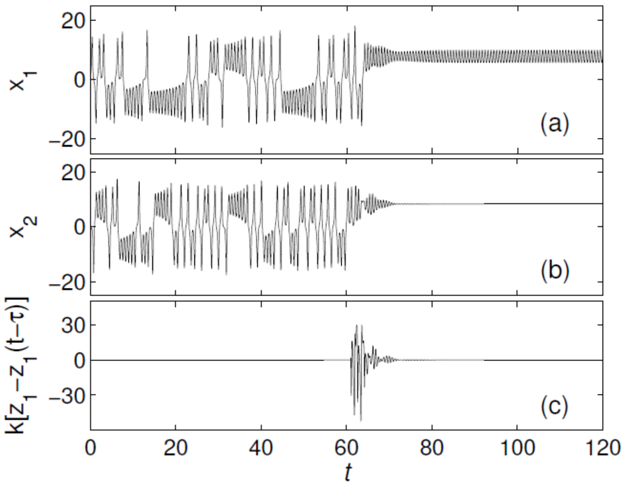

FIG. 2: Dynamics of (a) variable x_1, (b) variable x_2, and (c) the delayed coupling perturbation $k[z_1 - z_1(t-\tau)]$. The coupling control with $k = 2.5$ is switched on $t = 60$. The values of the parameters are $\rho_1 = 24.144$, $\rho_2 = 27$ and $\tau = 0.674$, and the connectivity matrix H is given by Eq. (7).

of Λ are all negative, so that both an UPO of Eq. (18a) and an USS of Eq. (18b) become stable. The parameters are given by $\rho_1 = 24.144$ (i.e., $\lambda_1 = -0.01792$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.0684$), $\tau = 0.674$, and Eqs. (7) and (19).

To verify the validity of the linear stability analysis, we have numerically investigated the original system Eqs. (1). The results of direct numerical integration of Eqs. (1) with z-coupling given by Eq. (7) are presented in Figure 2. Without the coupling ($k = 0$), two Lorenz systems demonstrate chaotic behaviors on the strange attractor. When the coupling perturbation of $k = 2.5$ is applied at $t = 60$, the $\mathbf{x}_1$-system and approaches an UPO, while the $\mathbf{x}_2$-system converges into an USS (Figure 2(a) and (b)). After a transient process, the coupling perturbation vanishes (Figure 2(c)) and thus our delay-coupling method allows for noninvasive control of UPO. It was observed in numerical simulations that the basin of attraction for stabilizing the UPO and USS include the whole area of the phase space in contrast to the situation of Hopf normal form system [22] and Lorenz system with unstable controller [11], which comes from the properties of strange attractor of two chaotic systems.

B. Stabilization of two USSs for $\rho_1 > \rho_H$ and $\rho_2 > \rho_H$

We now consider the problem for stabilizing USSs. First, the linear stability of USS of the reduced system (18) is analyzed. We linearize Eqs. (18) around $Z_1 = Z_2 = 0$ to obtain the characteristic equation for the eigenvalue Λ

$$\det\{A_0(\Lambda) - \Lambda I\} = 0, \tag{29}$$

where the linearized matrix A_0 is given by $A_0(\Lambda) = \begin{bmatrix} \lambda_1 + i\omega_1 & K_\beta \\ -K_\beta(1 - e^{-\Lambda\tau}) & \lambda_2 + i\omega_2 \end{bmatrix}$, I is the 2×2 identity matrix and $K_\beta = Ke^{i\beta}$. The bifurcation parameters λ_1 and λ_2 are positive and the delay time is chosen as $\tau = 2\pi/\bar{\omega}$ with $\bar{\omega} = (\omega_1 + \omega_2)/2$.

The matrix $A_0(\Lambda)$ remains invariant under the transformation $\Lambda \mapsto \Lambda + i(\omega_1 + \omega_2)/2$ due to $e^{i\bar{\omega}\tau} = 1$, and Eq. (29) can be rewritten in the form

$$(\Lambda - \lambda_1 + i\Delta\omega/2)(\Lambda - \lambda_2 - i\Delta\omega/2) + K_\beta^2(1 - e^{-\Lambda\tau}) = 0. \tag{30}$$

(Note that the stability is determined only by the real part of Λ.) This is a transcendental equation having an infinite number of roots and we are interested in the movement of the eigenvalues from the right half-plane for $K = 0$ into the left for a nonzero value of K.

Now, for a rough estimate, we again use an approximation $e^{-\Lambda\tau} \approx 1 - \Lambda\tau$ for $|\Lambda|\tau \ll 1$. Furthermore, if the diagonal coupling, $\beta = 0$, is assumed, then Eq. (30) is reduced, for $\omega_1 = \omega_2$ or $\lambda_1 = \lambda_2$, to a simple quadratic equation with

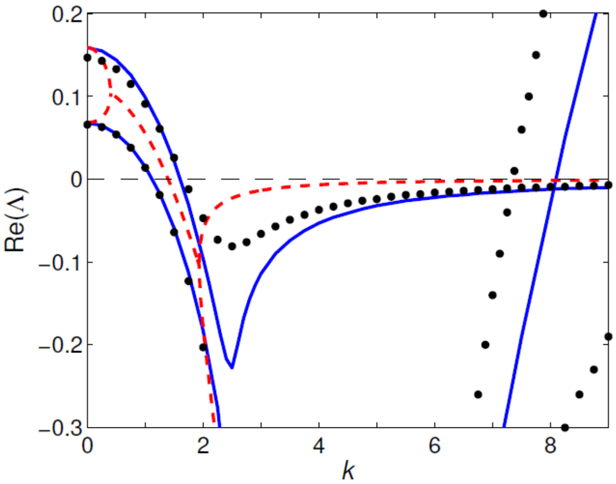

FIG. 3: (Color online) Real parts of the eigenvalues Λ of USSs in delay-coupled Lorenz systems as a function of coupling strength k. The dashed (red) and solid (blue) lines denote the solutions of the polynomial Eq. (31) and the transcendental Eq. (29), respectively. Dots correspond to Re Λ obtained from the exact characteristic Eqs. (33). Parameters are given by $\rho_1 = 30$ (i.e., $\lambda_1 = 0.159$ and $\omega_1 = 10.035$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.068$ and $\omega_2 = 10.579$), $\tau = 0.61$, and Eqs. (7) and (19).

the real coefficients as follows as

$$\Lambda^2 + (\kappa - \lambda_1 - \lambda_2)\Lambda + \lambda_1\lambda_2 = 0, \tag{31}$$

where $\kappa = K^2\tau$. Note that Eq. (31) coincides with Eq. (26) and yields the stability condition $K^2\tau > \lambda_1 + \lambda_2$. This means that stabilization of USS could also be explained with the same mechanism as the case of UPO, and the stability condition of USS reads $K^2\tau > \lambda_1 + \lambda_2$.

Next, we determine the *exact* eigenvalues Λ of the fixed points $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ by linearization of Eqs. (1) for $\rho_1,\rho_2 > \rho_H$:

$$\delta\dot{\mathbf{x}}_1 = \bar{A}_1\delta\mathbf{x}_1 + kH\delta\mathbf{x}_2, \tag{32a}$$

$$\delta\dot{\mathbf{x}}_2 = \bar{A}_2\delta\mathbf{x}_2 - kH\left(\delta\mathbf{x}_1 - \delta\mathbf{x}_{1,\tau}\right), \tag{32b}$$

which yield the characteristic equation as follow as

$$\det\{\bar{A}(\Lambda) - \Lambda I_6\} = 0, \tag{33}$$

where $\bar{A}_1 = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_1^*;\rho_1)$, $\bar{A}_2 = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_2^*;\rho_2)$, $\bar{A}(\Lambda) = \begin{bmatrix} \bar{A}_1 & kH \\ (e^{-\Lambda\tau}-1)kH & \bar{A}_2 \end{bmatrix}$, and I_6 is the 6×6 identity matrix.

Figure 3 shows the real parts of the eigenvalues Λ as a function of the coupling gain k, which were determined by different characteristic equations: The dashed line (red), solid line (blue) and dots (black) correspond to the eigenvalues obtained from the quadratic polynomial Eq. (31), the reduced transcendental Eq. (29) and the exact transcendental Eqs. (33), respectively. There exists an interval of coupling gain K for which the largest real parts of Λ are negative, so that two USSs become stable. The parameters are given by $\rho_1 = 30$ (i.e., $\lambda_1 = 0.159$ and $\omega_1 = 10.035$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.068$ and $\omega_2 = 10.579$), $\tau = 0.61$, and Eqs. (7) and (19).

Direct integration of the original system (1) with the above parameter values also confirms the results of linear analysis (Figure 4). Initially the decoupled system ($k = 0$) is in a chaotic regime, and the two USSs are stabilized when the coupling of $k = 2.5$ is switched on at $t = 40$ (Figure 2(a) and (b)). As seen from Figure 2(c), the perturbation vanishes as the stabilization of the USSs is attained. Therefore, our delay-coupling method allows for noninvasive control of dynamical systems. Note that the numerical simulations have revealed global characteristic of the basin of attraction for stabilizing the USSs, which is in contrast to the situation of Hopf normal form system [22].

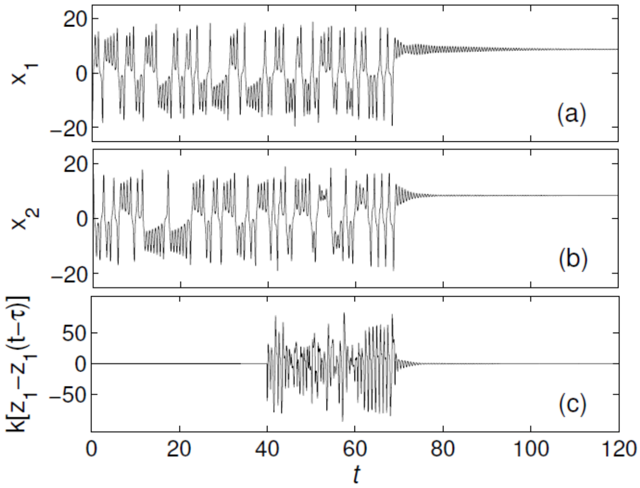

FIG. 4: Dynamics of (a) variable x_1, (b) variable x_2, and (c) the delayed coupling perturbation $k[z_1 - z_1(t-\tau)]$. The coupling control with $k = 2.5$ is switched on $t = 40$. The values of the parameters are $\rho_1 = 24.144$, $\rho_2 = 27$ and $\tau = 0.61$, and the connectivity matrix H is given by Eq. (7).

V. CONCLUSIONS

We have demonstrated a unified control method for stabilizing both a periodic orbit and a fixed point of the Lorenz system close to a subcritical Hopf bifurcation by noninvasive delayed coupling of two systems. We have developed systematic analytical approaches for reducing the system into Hopf normal forms and for stabilizing a periodic orbit and a fixed point using the multiple scales method and linear stability analysis. As a result the characteristic equations for Floquet exponents of the UPO and for eigenvalues of the USS have been derived in analytical form, which reveal the coupling parameters for successful stabilization.

To verify the validity of the linear stability analysis, we have performed numerical simulations of the original system, which show good agreement with the theory, i.e., the time-delay coupling method is capable of stabilizing not only UPOs but USSs as well in the Lorenz systems for a wide interval of the coupling strength. In particular, two Lorenz systems with this coupling exhibit a global basin of attraction for stabilizing an UPO and an USS at the same time due to the nature of the strange attractor, which is in striking contrast to the situation of the delay-coupled Hopf normal form systems [22] and Lorenz system with unstable controller [11].

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[1] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[2] K. Pyragas, Philos. Trans. R. Soc. London, Ser. A **364**, 2309 (2006).
[3] *Handbook of Chaos Control*, 2nd ed, edited by E. Schöll and H. G. Schuster (Wiley-VCH, Weinheim, 2008).
[4] H. Ando, K. Takehara, and M. U. Kobayashi, Phys. Rev. E **96**, 012148 (2017).
[5] M. Zeitz, P. Gurevich, and H. Stark, Eur. Phys. J. E **38**(3), 22 (2015).
[6] V. Pyragas and K. Pyragas, Commun Nonlinear Sci Numer Simulat **73**, 338-350. (2019).
[7] H. Nakajima, Phys. Lett. A **232**, 207 (1997).
[8] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, Phys. Rev. Lett. **78**, 203 (1997).
[9] H. Nakajima and Y. Ueda, Phys. Rev. E **58**, 1757 (1998).
[10] K. Pyragas Phys. Rev. Lett. **86**, 2265 (2001).
[11] V. Pyragas and K. Pyragas, Phys. Rev. E **73**, 036215 (2006).
[12] B. Fiedler, V. Flunkert, M. Georgi, P. Hövel, and E. Schöll, Phys. Rev. Lett. **98**, 114101 (2007).
[13] C. M. Postlethwaite and M. Silber, Phys. Rev. E **76**, 056214 (2007).
[14] C.-U. Choe, T. Dahms, P. Hövel, and E. Schöll, Phys. Rev. E **81**, 025205(R) (2010).
[15] C.-U. Choe, H. Jang, P. Hövel, and E. Schöll, *Stabilization of Periodic Orbits in Networks of subcritical Hopf-bifurcation System with Delayed Coupling* (2011) in preparation.
[16] K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **89**, 244103 (2002).
[17] K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. E **70**, 026215 (2004).
[18] P. Hövel and E. Schöll, Phys. Rev. E **72**, 046203 (2005).
[19] S. Yanchuk, M. Wolfram, P. Hövel, and E. Schöll, Phys. Rev. E **74**, 026201 (2006).
[20] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998).
[21] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Physica D **129**, 15 (1999).
[22] C.-U. Choe, V. Flunkert, P. Hövel, H. Benner, and E. Schöll, Phys. Rev. E **75**, 046206 (2007).
[23] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[24] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (Springer, New York, 1982).
[25] A. H. Nayfeh and B. Balachandran, *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods* (Wiely, New York, 1995)
[26] K. Pyragas, Phys. Rev. E **66**, 026207 (2002).