The International Journal of Engineering and Science (IJES)
|| Volume || 12 || Issue || 1 || January || Pages || PP 33-46 || 2023 ||
ISSN (e): 2319-1813 ISSN (p): 20-24-1805

Analysis and Evaluation of Real-valued Functions in Mathematical Morphology

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---***ABSTRACT***---
The mathematical methods offered by mathematical morphology are mainly oriented towards problems in image or signal processing and analysis as well as other fields such as artificial intelligence, pattern recognition, and soft computing. Since mathematical morphology is a combined geometric and algebraic framework, its basic operations can be defined on sets and numerical functions whenever their underlying algebraic structure is a complete lattice. The fundamental idea behind the morphological approach is to transform a given set or a function by means of simple structuring elements into another set or function that preserves the essential chracteristics of the source set or function in such a way as to make easier its analysis or interpretation in the case of real world applications. In general, the structuring elements are also sets of smaller extension or functions defined on a finite subdomain with respect to the original domain. In this work we give a brief theoretical foundation of the basic operations in mathematical morphology with emphasis on the algebraic determination and numerical evaluation for the case of real valued functions.

KEYWORDS: *lattice algebra, lattice computing, mathematical morphology, numerical function evaluation*

Date of Submission: 03-01-2023 Date of Acceptance: 16-01-2023

I. INTRODUCTION

Mathematical morphology [1-5] is a non-linear approach to image processing and analysis [6,7] with a mathematical foundation that entails both a geometrical as well as an algebraic aspect as a result of conceptually using complete lattices whose elements can be sets or numerical functions. The central idea of the morphological approach is to process a digital image using as a scanning element a smaller image with a predetermined geometry adequate for the evaluation of the geometrical or topological characteristics of objects contained in the source image. The smaller image is commonly known as a structuring element. Thus, the fundamental purpose is to transform a given image into another more appropriate for its analysis or data understanding based on the resulting interaction with the chosen structuring element [8-10].

Mathematical morphology was created in the middle 19’sixties in France, after George Matheron studied the relationship between the geometry of porous media and their permeabilities and, at the same time, Jean Serra quantified the petrography of iron minerals with the objective of predicting their grinding properties. The aforementioned studies conducted both scientists to establish the theoretical grounds for the analysis of binary images. A porous medium is binary in the sense that a given point belongs to a pore or to the material that surrounds it. Thus, the material surrounding the pores can be taken as a set and all pores would be the complement set. Therefore, objects in a binary image can be treated with set operations. In 1967, Matheron proposed the first morphological transformations for finding the geometry in binary images.

A major portion of mathematical morphology was developed in the Center of Mathematical Morphology (Centre de Morphologie Mathématique) created in 1968 as part of the Paris School of Mines at Fontainebleau and directed by Matheron and Serra [11]. The development of specialized equipment for image processing, such as the Texture Analyzer, allowed the use of new transformations that would adapt to the type of problem under scrutiny. Hence, mathematical morphology was elaborated by fusioning the theoretical aspects together with real world applications and algorithmic designs. For almost a decade mathematical morphology dealt only with binary images considered as sets and since middle 19’seventies it was developed to process and analyze grayscale images by extending the erosion and dilation operations to numerical functions [12]. By the end of the 19’eighties and beginning of the 19’ninties, mathematical morphology was founded on a more general algebraic view based on complete lattices and digital spaces [13-15].

The naïve idea that we have about the structure of objects in a scene is not enough since it can not always be precised. Furthermore, it is practically imposible to give an objective and complete description of an arbitrary object. An observer will see an object in a personal manner emphasizing certain characteristics that are of interest to him and so will transform the object into another one with certain details highlighted. The observer could say that part of the object that captures his attention, have spikes, is squared or rounded, small or big. The attributes perceived by the observer are the result that he may have by comparing part of the object under study with a star, a square, a circle, or other similar objects in shape and size. These last objects can be indentified as structuring elements and the way of relating them perceptually with the object as a morphological transformation. Thus, mathematical morphology is a theory that considers the local geometrical traits of specific interest when analyzing one or more objects. Currently, mathematical morphology is considered a versatile tool for signal and image processing and analysis, specially in those applications where the geometrical aspects are relevant [9-10].

The following work presents the elementary theory of mathematical morphology as a body of knowledge by giving proofs of several basic properties and of some important theorems relative to the fundamental morphological operations of erosion, dilation, opening and closing for sets and functions. As a complement to our theoretical exposition we illustrate with simple examples how to determine algebraically as well as to evaluate numerically the morphological erosion and dilation with real valued functions [16].

Preliminary Concepts

The following background concepts of a geometrical nature are fundamental for the definitions of the basic operations in mathematical morphology.

Definition 1. *Let $A \subseteq \mathbb{R}^n$, the translation of A by vector $x \in \mathbb{R}^n$, is the set given by $A_x = \{a + x : a \in A\}$, and the symmetric of A is the set, $\hat{A} = \{-a : a \in A$, where symmetry is realized with respect to the origin.*

The next proposition list several equalities that relate the concepts of translation and symmetry with the set operations of union, intersection, and complementation. In item b) of **Proposition 1**, we will write B_x^c instead of $(B_x)^c$ to simplify notation.

Proposition 1. *Valid identities between set operations with translations and symmetrics*:

a) $\bigcup_{b\in B} A_b = \bigcup_{a\in A} B_a$ *(commutativity of translation with union),*
b) $(B^c)_x = (B_x)^c$ *(commutativity of complement with translation),*
c) $\bigcap_{b\in B} A_b = \bigcap_{a\in A^c} (B^c)_a$ *(equivalence of intersection of translations by complementation),*
d) $(A_x)_y = A_{x+y}$ *(composition of translations),*
e) $(\bigcup_{b\in B} A_b)_x = \bigcup_{b\in B} A_{b+x}$ *(distributivity of translation over union),*
f) $\bigcup_{x\in X} (\bigcup_{y\in Y} B_y)_x = \bigcup_{y\in Y} (\bigcup_{x\in X} B_x)_y$ *(conmutativity betwee double unions and translations),*
g) $\widehat{\bigcup_{b\in B} A_b} = \bigcup_{b\in B} \widehat{A_b}$ *(distributivity of symmetric over union), and*
h) $\widehat{B_x} = \hat{B}_{-x}$ *(opposite of a translation by symmetrization).*

Proof :

a) $x \in \bigcup_{b\in B} A_b \Leftrightarrow \exists a \in A, b \in B$ such that $x = a + b \Leftrightarrow \exists b \in B, a \in A$ such that $x = b + a \Leftrightarrow x \in \bigcup_{a\in A} B_a$,

b) $y \in (B^c)_x \Leftrightarrow y = b' + x$ such that $b' \in B^c \Leftrightarrow y \neq b + x \;\; \forall b \in B \Leftrightarrow y \in (B_x)^c$,

c) $\bigcap_{b\in B} A_b = (\bigcup_{b\in B} (A^c)_b)^c = (\bigcup_{a\in A^c} B_a)^c = \bigcup_{a\in A^c} (B^c)_b$,

d) $z \in (A_x)_y \Leftrightarrow z = (a + x) + y,\ a \in A \Leftrightarrow z = a + (x + y),\ a \in A \Leftrightarrow z \in A_{x+y}$,

e) $y \in (\bigcup_{b\in B} A_b)_x \Leftrightarrow \exists a \in A, b \in B$ such that $y = (a + b) + x \Leftrightarrow \exists a \in A, b \in B$ such that $y = a + (b + x) \Leftrightarrow \bigcup_{b\in B} A_{b+x}$,

f) $z \in \bigcup_{x\in X} (\bigcup_{y\in Y} B_y)_x \Leftrightarrow \exists x \in X, y \in Y$ such that $z = (b + y) + x \Leftrightarrow \exists y \in Y, x \in X$ such that $z = (b + x) + y \Leftrightarrow z \in \bigcup_{y\in Y} (\bigcup_{x\in X} B_x)_y$,

g) $\widehat{\bigcup_{b\in B} A_b} = \{-(a + b) : a \in A \text{ y } b \in B\} = \bigcup_{b\in B} \widehat{A_b}$, and

h) $\widehat{B_x} = \{-(b+x): b \in B\} = \{-b-x: -b \in \hat{B}\} = \hat{B}_{-x}$.

II. PARTIAL ORDERINGS AND ALGEBRAIC LATTICES

Definitions of the morphological mathematical operations assume that the set used as workspace is endowed with the algebraic structure of a complete lattice since the corresponding operations entail that structure in a natural way. Therefore in this Section we present the concepts of partial orderings and lattices including several of their properties and types of lattices [17,18].

Definition 2. *A is a partially ordered set if given a binary relation $\preccurlyeq$ defined on A, satisfies the following properties for any $x, y, z \in A$: a) $x \preccurlyeq x\ \forall\ x \in A$ (reflexivity), b) $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$ (transitivity), and c) $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$ (antisimmetry). The expression $x \preccurlyeq y$ reads "x precedes y".*

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We write a partially ordered set or a partial ordering to be brief, as the pair of objects $(A, \preccurlyeq)$, highlighting the fact that A is the set on which the binary relation $\preccurlyeq$ is defined. Two elements of A are *comparable*, if and only if, $a \preccurlyeq b$ or $b \preccurlyeq a$, otherwise a and b *are not* comparable. The word "partial" is used to emphasize that some pairs of elements in A are not comparable. Let $(A, \preccurlyeq)$ be a partial ordering. The elements a, b are called an *upper bound* or *lower bound* of A, respectively, if $x \preccurlyeq a$ or $b \preccurlyeq x$ for all $x \in A$. The lowest of all upperbounds is known as the *supremum* of A, denoted by $\sup(A)$, and the greatest of all lower bounds is known as the *infimum* of A symbolized by $\inf(A)$. If $\sup(A) \in A$, then $\sup(A)$ is called the *maximum* of A denoted by $\max(A)$; whereas if $\inf(A) \in A$, then it is named the *minimum* of A written as $\min(A)$. In a partially ordered set A the binary relation $\preccurlyeq$ can be described in the following way. The element a is an *immediate predecessor* of b in A or b is an *immediate succesor* of a in A, expressed as $a \ll b$, if no element x exists such that $a \prec x \prec b$.

Definition 3. *Let L be a non-empty partially ordered set. A lattice, denoted as $(L, \wedge, \vee)$, is an algebraic structure equipped with two binary operations named "meet" and "join", symbolized respectively by $\wedge$ and $\vee$, for which $a \wedge b = \inf(a, b)$ and $a \vee b = \sup(a, b)$ exist and belong to L for each pair of elements $a, b \in L$.*

Definition 4. *L endowed with the binary operations $\wedge$ and $\vee$ is a lattice algebra if for any $a, b, c \in L$ the following laws are verified: a) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ (commutativity), b) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$ (associativity), and c) $a \wedge (a \vee b) = a$ y $a \vee (a \wedge b) = a$ (absorption).*

We will see that **Definitions 3** and **4** are equivalent since both operations, $\wedge$ and $\vee$, verify the properties of a lattice algebra. Similarly, given a lattice algebra a partial ordering is defined as follows: $x \preccurlyeq y \Leftrightarrow x \wedge y = x$, or equivalently, $x \preccurlyeq y \Leftrightarrow x \vee y = y$.

Proof : First we check that $(L, \preccurlyeq)$ is a partial ordering: a) to prove the reflexivity condition used is made of the absorption property so that, $x \wedge x = x \wedge (x \vee (x \wedge y)) = x$, which implies by definition that $x \preccurlyeq x$; b) for transitivity, suppose that $x \preccurlyeq y$ and also that $y \preccurlyeq z$, then, $x \wedge y = x$ and $y \wedge z = y$. Thus, $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$, and by definition, $x \preccurlyeq z$, and c) antisimmetry, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then, $x \wedge y = x$ y $y \wedge x = y$. By the commutative property results that, $x = x \wedge y = y \wedge x = y$, henceforth, $x = y$. On the other hand, considering L as a lattice algebra, we see that $(x \wedge y) \wedge x = x \wedge (y \wedge x) = x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y$. In addition, $(x \wedge y) \wedge y = x \wedge (y \wedge y) = x \wedge y \Leftrightarrow x \wedge y \preccurlyeq x$ y $x \wedge y \preccurlyeq y$. Now, assuming that $z \preccurlyeq x$ and $z \preccurlyeq y$, then $z \wedge x = z$ y $z \wedge y = z$. Consequently, $(z \wedge x) \wedge (z \wedge y) = z \wedge z = z$, which is equivalent to, $z \wedge (x \wedge y) = z$ and therefore, $z \preccurlyeq x \wedge y$. Reciprocally, if $z \preccurlyeq x \wedge y$, and since $x \wedge y \preccurlyeq x$ y $x \wedge y \preccurlyeq y$, by transitivity we obtain that, $z \preccurlyeq x$ and $z \preccurlyeq y$. Thus, as shown, a lattice algebra L is a lattice ■

In this work, we restrict the type of algebraic structures to bounded and complete lattices defined next. If a lattice L has a maximum element and a minimum element, it is called a *bounded lattice* which we write in full as $(L, \wedge, \vee, \max(L), \min(L))$. Also, a lattice L in which every non-empty subset has an infimum and a supremum, is said to be a *complete lattice*; in particular, every complete lattice is bounded.

III. MATHEMATICAL MORPHOLOGY OF SETS

The operations of mathematical morphology allow for the extraction of possible relevant geometrical structures of the objects contained in a given digital image by considering it as a subset of $\mathbb{R}^2$ or of $\mathbb{Z}^2$. The extraction is achieved by probing the corresponding subset with another set acting as a structuring element or SE for short, whose geometrical shape and size depends on the information required for the analysis of objects that appear in the given image. We remark that in image processing applications a structuring element is taken as a much smaller image in size with respect to the original one.

The basic morphological operations have been proposed to allow the SE to interact with the objects of interest in a given scene by overlaying it point by point and probe if the SE "touches" or "fits within" those objects. The structuring elements used for exploring two dimensional images are known as *flat structuring elements* since they have the same dimension as the image under study, they depend only on its domain and do not depend on gray scale values. However, three-dimensional structuring elements are known as *volumetric* SE's or *none-planar* SE's and must have values in the same dynamic range as the original image.

In each structuring element it is required to fix one of its elements as a reference or pivot point, usually its center, to allow for changing the way a morphological operation acts. Hence, upon geometrical translation or displacement of the SE over the image, its center is positioned on the pixel scanned. Once more, the geometric shape and size of a SE must be adapted to the image object characteristics; for example, a rectangular shaped SE is adequate for extracting objects whose forms are composed of rectangles. In what follows, we define the fundamental morphological operations for processing and analyzing binary images. The sets considered in the corresponding definitions are subsets of the Euclidean *n*-dimensional real space, i. e., $\mathbb{R}^n$.

Erosion and Dilation of Sets

Definition 5. *The erosion of set A by the structuring element B, symbolized by $A \ominus B$, is defined as,*

$$A \ominus B = \{x : B_x \subseteq A\} ,$$

where B_x is the translation of set B to point $x \in A$.

The result obtained with erosion is a new set containing those points $x \in \mathbb{R}^n$ for which B is contained in A (is a proper subset) when its center is positioned or displaced at x.

Theorem 1. *The erosion operation between sets A and B can be expressed as,*

$$A \ominus B = \bigcap_{b \in B} A_{-b} .$$

Proof: $x \in A \ominus B \Leftrightarrow B_x \subseteq A \Leftrightarrow b + x = y \in A \quad \forall b \in B \Leftrightarrow x = y - b$ such that $y \in A \quad \forall b \in B \Leftrightarrow x \in A_{-b} \quad \forall b \in B \Leftrightarrow x \in \bigcap_{b \in B} A_{-b}$ ■

Definition 6. *The dilation of set A by the structuring element B, symbolized by $A \oplus B$, is defined as,*

$$A \oplus B = \{x : \hat{B}_x \cap A \neq \emptyset\} .$$

The dilation operation gives another set whose points $x \in \mathbb{R}^n$ are such that the center of $\hat{B}$ (symmetric of B) upon translation to x verifies that $\hat{B}_x \cap A \neq \emptyset$ (non-void intersection), meaning that $\hat{B}_x$ "touches" or "hits" A.

Theorem 2. *The dilation operation between sets A and B can be expressed as,*

$$A \oplus B = \bigcup_{b \in B} A_b .$$

Proof: $x \in A \oplus B \Leftrightarrow \hat{B}_x \cap A \neq \emptyset \Leftrightarrow \exists y$ such that $y \in \hat{B}_x$ and $y \in A \Leftrightarrow y = x - b$ for some $b \in B$ and $y \in A \Leftrightarrow x = y + b$ for some $b \in B$ and $y \in A \Leftrightarrow x \in A_b$ for some $b \in B \Leftrightarrow x \in \bigcup_{b \in B} A_b$ ■

Proposition 2. *The distributivity of the symmetric is valid for the dilation operation, i. e., $\widehat{A \oplus B} = \hat{A} \oplus \hat{B}$.*

Proof :

$$\widehat{\mathrm{A} \oplus \mathrm{B}} = \widehat{\bigcup_{b \in B} A_b} = \bigcup_{b \in B} \widehat{A_b} = \bigcup_{b \in B} \hat{A}_{-b} = \bigcup_{b \in \hat{B}} \hat{A}_b = \hat{A} \oplus \hat{B} \ ■$$

Note the use of identity g) of Proposition 1 in passing from the first to the second equality.

Opening and Closing of Sets

From the basic morphological operations of erosion and dilation new operations can be constructed. For example, the morphological opening and closing operations can be built by chaining them algebraically as is explained next.

Definition 7. *The opening of set A by the structuring element B, denoted by $A \circ B$, is defined as the following algebraic composition, $A \circ B = (A \ominus B) \oplus B$.*

In words, the morphological opening consists of applying the erosion of A by B followed by the dilation with B of the eroded set $A \ominus B$. This new operation selects points belonging to A that are covered by some translation of the SE B that is included in A. Thus, the opening of A is obtained by moving B *inside* A, even tangentially to the border or frontier of A without any element of B staying outside A. The following theorem corresponds to the given verbal description.

Teorema 3. *The opening operation between sets A and B can be expressed as,*

$$A \circ B = \bigcup_{B_x \subseteq A} B_x \,.$$

Proof: $x \in A \circ B \Leftrightarrow x \in (A \ominus B) \oplus B \Leftrightarrow \hat{B}_x \cap (A \ominus B) \neq \emptyset \Leftrightarrow \exists y$ such that $y \in \hat{B}_x$ and $y \in (A \ominus B)$ $\Leftrightarrow \exists y, \exists b \in B$ such that $y = x - b$ and $B_y \subseteq A \Leftrightarrow \exists y, \exists b \in B$ such that $x = y + b$ and $B_y \subseteq A \Leftrightarrow \exists B_y \subseteq A$ such that $x \in B_y \Leftrightarrow x \in \bigcup_{By \subseteq A} B_y$.

Definition 8. *The closing of A by the structuring element B, denoted by $A \bullet B$, is defined as the following algebraic composition, $A \bullet B = (A \oplus B) \ominus B$.*

In words, the morphological closing consists of applying the dilation A by B followed by the erosion with B of the dilated set $A \oplus B$. The theorem below shows that the closing is equivalent to the intersection of the complements of all translated copies at x of the symmetrical SE contained in the complement of A.

Theorem 4. *The closing operation between sets A and B can written as,*

$$A \bullet B = \left(\bigcup_{\hat{B}_x \subseteq A^c} \hat{B}_x \right)^c = \bigcap_{\hat{B}_x \subseteq A^c} \hat{B}_x^c \,.$$

.

Proof : $x \in A \bullet B \Leftrightarrow x \in (A \oplus B) \ominus B \Leftrightarrow B_x \subseteq A \oplus B \Leftrightarrow (A \oplus B)^c \subseteq B_x^c \Leftrightarrow (\forall y, y \in (A \oplus B)^c \Rightarrow y \in B_x^c)$ $\Leftrightarrow (\forall y, \hat{B}_y \cap A = \emptyset \Rightarrow \forall b \in B, y \neq x + b) \Leftrightarrow (\forall y, \hat{B}_y \subseteq A^c \Rightarrow \forall b \in B, x \neq y - b) \Leftrightarrow (\forall y, \hat{B}_y \subseteq A^c \Rightarrow x \notin \hat{B}_y) \Leftrightarrow (\forall y, \hat{B}_y \subseteq A^c \Rightarrow x \in \hat{B}_y^c) \Leftrightarrow x \in \hat{B}_y^c, \ \forall \hat{B}_y \subseteq A^c \Leftrightarrow x \in \bigcap_{\hat{B}_y \subseteq A^c} \hat{B}_y^c$.

The morphological opening removes points of A where the translations of the SE do not fit locally, whereas the morphological closing adds those points not covered by translations of $\hat{B}$ included in the complement of A. The core idea of the opening and closing operations consists in rebuilding in a global manner the initial shape of the eroded or dilated set respectively.

IV. MATHEMATICAL MORPHOLOGY OF FUNCTIONS

In this section additional concepts are introduced in order to extend the mathematical morphology of sets to include the case of real valued functions. The extension is achieved by associating a unique set to a given function. In what follows, the domain of a real function f of one or more real variables is a subset of $\mathbb{R}^{n-1}$ for $n \geq 2$, i. e., $D_f \subseteq \mathbb{R}^{n-1}$.

Definition 9. *Let $a \in \mathbb{R}^{n-1}$ be a vector and $f: D_f \to \mathbb{R}$ a function. The translation of f by a denoted by f_a, is defined as the function $f_a: D_{f_a} \to \mathbb{R}$, where $f_a(x) = f(x-a)$. Besides, the symmetric of f written $\hat{f}$, is defined as the function $\hat{f}: \widehat{D_f} \to \mathbb{R}$, where $\hat{f}(x) = f(-x)$.*

Definition 10. *Let $A \subseteq \mathbb{R}^n$, $D_A = \{x \in \mathbb{R}^{n-1}: \exists y \in \mathbb{R}$ such that $(x,y) \in A\}$, and $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ the extended real number system. The top of A is a function denoted by $\mathcal{T}(A): D_A \to \mathbb{R}^*$ such that $\mathcal{T}(A)(x) = \sup\{y: (x,y) \in A\}$.*

Definition 11. *Let $f: D_f \to \mathbb{R}$ be a function. The umbra or shadow of f is defined as the set, $\mathcal{S}(f) = \{(x,y) \in D_f \times \mathbb{R}: y \le f(x)\}$.*

Lemma 1. *Let $f: D_f \times \mathbb{R} \to \mathbb{R}$ be a function, then, $\mathcal{T}(\mathcal{S}(f)) = f$.*

Proof: First it is shown that the domains of the functions f and $\mathcal{T}(\mathcal{S}(f))$ are the same. If $x \in D_{\mathcal{S}(f)} \Rightarrow \exists y \in \mathbb{R}$ such that $(x,y) \in \mathcal{S}(f) \Rightarrow x \in D_f$. On the other way, if $x \in D_f$ and $y = f(x) \Rightarrow y \le f(x) \Rightarrow \exists y \in \mathbb{R}$ such that $(x,y) \in \mathcal{S}(f) \Rightarrow x \in D_{\mathcal{S}(f)} \Rightarrow D_{\mathcal{S}(f)} = D_f$. Second we prove that, $\mathcal{T}(\mathcal{S}(f))(x) = f(x)\ \forall x \in D_f$. Specifically, $\mathcal{T}\big(\mathcal{S}(f)\big)(x) = \sup\{y: (x,y) \in \mathcal{S}(f)\} = \sup\{y: y \le f(x)\} = f(x) \Rightarrow \mathcal{T}\big(\mathcal{S}(f)\big) = f$ ■

Definition 12. *Let I be an index set and $\mathcal{F} = \left\{f_i \middle| f_i: D_{f_i} \to \mathbb{R}^*, i \in I\right\}$ be a family of real functions. We define the infimum and supremum functons of $\mathcal{F}$, written respectively as $\wedge\mathcal{F}: D_\wedge \to \mathbb{R}^*$ and $\vee\mathcal{F}: D_\vee \to \mathbb{R}^*$, by the expressions:*

$$\text{a) } \wedge\mathcal{F} = \left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x) = \inf_{i\in I}\{f_i(x)\} \ \text{ where } D_\wedge = \bigcap\nolimits_{i\in I} D_{f_i},$$

$$\text{b) } \vee\mathcal{F} = \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x) = \sup_{i\in I}\{f_i(x)\} \text{ where } D_\vee = \bigcup\nolimits_{i\in I} D_{f_i}.$$

Lemma 2 *Let the family $\mathcal{F}$ of real functions f_i indexed by I be equipped with the operations of infimum and supremum, and let $\mathcal{P}(\mathbb{R}^n)$ be the family of subsets of $\mathbb{R}^n$ with the operations of intersection and union. Then, the shadow is a homomorphism between the complete lattices ($\mathcal{F}$,∧,∨) and ($\mathcal{P}(\mathbb{R}^n)$,∩,∪), i. e., the following equalities are satisfied:*

$$\mathcal{S}\left(\bigwedge_{i\in I} f_i\right) = \bigcap\nolimits_{i\in I} \mathcal{S}(f_i) \ \ \textit{and} \ \ \mathcal{S}\left(\bigvee_{i\in I} f_i\right) = \bigcup\nolimits_{i\in I} \mathcal{S}(f_i).$$

Proof : Let $f, g \in \mathcal{F}$ then,

a) $\mathcal{S}(\wedge_{i\in I} f_i) = \{(x,y) \in D_\wedge \times \mathbb{R} : y \le (\wedge_{i\in I} f_i)(x)\}$
$= \{(x,y) \in D_\wedge \times \mathbb{R} : y \le \wedge_{i\in I} f_i(x)\}$
$= \{(x,y) \in D_\wedge \times \mathbb{R} : y \le f_i(x)$ for all $i \in I\}$
$= \cap_{i\in I} \ \{(x,y) \in D_\wedge \times \mathbb{R} : y \le f_i(x)\} = \cap_{i\in I} \mathcal{S}(f_i)$ and

b) $\mathcal{S}(\vee_{i\in I} f_i) = \{(x,y) \in D_\vee \times \mathbb{R} : y \le (\vee_{i\in I} f_i)(x)\}$
$= \{(x,y) \in D_\vee \times \mathbb{R} : y \le \vee_{i\in I} f_i(x)\}$
$= \{(x,y) \in D_\vee \times \mathbb{R} : y \le f_i(x)$ for some $i \in I\}$
$= \cup_{i\in I} \ \{(x,y) \in D_\vee \times \mathbb{R} : y \le f_i(x)\} = \cup_{i\in I} \mathcal{S}(f_i)$.

Erosion and Dilation of Functions

Definition 13. *The erosion function of $f, g \in \mathcal{F}$, denoted by $f \ominus g: D_{f\ominus g} \to \mathbb{R}$ is defined as the top of the erosion of their corresponding shadow sets, symbolically, $f \ominus g = \mathcal{T}(\mathcal{S}(f) \ominus \mathcal{S}(g))$.*

Theorem 5. *The erosion of f by g is equivalent to the expression given by,*

$$(f \ominus g)(x) = \bigwedge_{a \in D_g} \{f(x+a) - g(a)\} \ \ where \ \ (x+a) \in D_f \ \ and \ \ D_{f \ominus g} = D_f \ominus D_g.$$

.

Proof: Changing variables, let $x = x' - a$ and $y = y' - b$, hence we have that,

$$\{(x', y') - (a, b) : y' \leq f(x'), x' \in D_f\} = \{(x, y) : y \leq f(x+a) - b, (x+a) \in D_f\}.$$

In what follows we omit the fact that $(x+a) \in D_f$ with the purpose of abbreviating intermediate expressions and at the end we restitute this condition. For all $b \leq g(a)$, we have the following set inclusion,

$$\{(x, y) : y \leq f(x+a) - g(a)\} \subseteq \bigcap_{b \leq g(a)} \{(x, y) : y \leq f(x+a) - b\}$$

$$\Rightarrow \bigcap_{b \leq g(a)} \{(x, y) : y \leq f(x+a) - b\} = \{(x, y) : y \leq f(x+a) - g(a)\}.$$

.

Next, by considering our previous definitions and results, we can see that,

$$\mathcal{S}(f) \ominus \mathcal{S}(g) = \bigcap_{(a,b) \in \mathcal{S}(g)} \mathcal{S}(f)_{(a,b)} = \bigcap_{a \in D_g, b \leq g(a)} \{(x, y) - (a, b) : y \leq f(x)\}$$

$$= \bigcap_{a \in D_g} \bigcap_{b \leq g(a)} \{(x, y) : y \leq f(x+a) - b\} = \bigcap_{a \in D_g} (\{(x, y) : y \leq f(x+a) - g(a)\})$$

$$= \bigcap_{a \in D_g} \mathcal{S}[f_{-a} - g(a)] = \mathcal{S}\left[\bigwedge_{a \in D_g} \left(f_{-a} - g(a)\right)\right],$$

where the last equality follows from **Lemma 2**. Thus,

$$f \ominus g = \mathcal{T}(\mathcal{S}(f) \ominus \mathcal{S}(g)) = \mathcal{T}\left(\mathcal{S}\left[\bigwedge_{a \in D_g} (f_{-a} - g(a))\right]\right) = \bigwedge_{a \in D_g} (f_{-a} - g(a))$$

$$\Rightarrow (f \ominus g)(x) = \bigwedge_{a \in D_g} \{f(x+a) - g(a)\} \text{ where } (x+a) \in D_f \, .$$

Finally, by **Definition 12** and **Theorem 1**, it turns out that, $D_{f \ominus g} = \cap_{a \in D_g} (D_f)_{-a} = D_f \ominus D_g$ ■

Corollary 1. *The shadow of the erosion of* $f, g \in \mathcal{F}$ *is given by* $\mathcal{S}(f \ominus g) = \mathcal{S}(f) \ominus \mathcal{S}(g)$.

Proof: From the final equations in the proof of **Theorem 5** we see that,

$$\mathcal{S}(f) \ominus \mathcal{S}(g) = \mathcal{S}\left[\bigwedge_{a \in D_g} \left(f_{-a} - g(a)\right)\right] = \mathcal{S}(f \ominus g) \ ■$$

Definition 14. *The dilation function of* $f, g \in \mathcal{F}$*, denoted by,* $f \oplus g : D_{f \oplus g} \to \mathbb{R}$*, is defined as the top of the dilation of their corresponding shadows sets, symbolically,* $f \oplus g = \mathcal{T}(\mathcal{S}(f) \oplus \mathcal{S}(g))$.

Theorem 6. *The dilation of* f *by* g *is equivalent to the expression given by,*

$$(f \oplus g)(x) = \bigvee_{a \in D_g} \{f(x-a) + g(a)\} \ \ where \ \ (x-a) \in D_f \ \text{ and } D_{f \oplus g} = D_f \oplus D_g.$$

Proof: Let $x = x' + a$ and $y = y' + b$ be new variables, thus we can see that,

$$\{(x',y')+(a,b): y' \leq f(x'), x' \in D_f\} = \{(x,y): y \leq f(x-a)+b, (x-a) \in D_f\}.$$

.
Again, we dismiss the fact that $(x-a) \in D_f$ to shorten intermediate expressions and at the end we point out the same condition. For all $b \leq g(a)$ we have the following subset relation,

$$\bigcup_{b \leq g(a)} \{(x,y): y \leq f(x-a)+b\} \subseteq \{(x,y): y \leq f(x-a)+g(a)\}$$
$$\Rightarrow \bigcup_{b \leq g(a)} \{(x,y): y \leq f(x-a)+b\} = \{(x,y): y \leq f(x-a)+g(a)\}\,.$$

.
From our previous definitions and results, we have that,

$$\mathcal{S}(f) \oplus \mathcal{S}(g) = \bigcup_{(a,b)\in\mathcal{S}(g)} \mathcal{S}(f)_{(a,b)} = \bigcup_{a\in D_g, b\leq g(a)} \{(x,y)+(a,b): y \leq f(x)\}$$

$$= \bigcup_{a\in D_g} \bigcup_{b\leq g(a)} \{(x,y): y \leq f(x-a)+b\} = \bigcup_{a\in D_g} (\{(x,y): y \leq f(x-a)+g(a)\})$$

$$= \bigcup_{a\in D_g} \mathcal{S}[f_a + g(a)] = \mathcal{S}\left[\bigvee_{a\in D_g} (f_a + g(a))\right],$$

where the last equality follows from **Lemma 2**. Therefore,

$$f \oplus g = \mathcal{T}(\mathcal{S}(f) \oplus \mathcal{S}(g)) = \mathcal{T}\left(\mathcal{S}\left[\bigvee_{a\in D_g} (f_a + g(a))\right]\right) = \bigvee_{a\in D_g} (f_a + g(a))$$

$$\Rightarrow (f \oplus g)(x) = \bigvee_{a\in D_g} \{f(x-a)+g(a)\} \text{ where } (x-a) \in D_f.$$

Also, from **Definition 12** and **Theorem 2**, it follows that, $D_{f\oplus g} = \bigcup_{a\in D_g} (D_f)_a = D_f \oplus D_g$ ■

Corollary 2. *The shadow of the dilation of* $f, g \in \mathcal{F}$ *is given by* $\mathcal{S}(f \oplus g) = \mathcal{S}(f) \oplus \mathcal{S}(g)$.

Proof: From the final equations in the proof of Theorem 6 we conclude that,

$$\mathcal{S}(f) \oplus \mathcal{S}(g) = \mathcal{S}\left[\bigvee_{a\in D_g} \big(f_a + g(a)\big)\right] = \mathcal{S}(f \oplus g) \;■$$

Opening and Closing of Functions

Definition 15. *Let* $f, g \in \mathcal{F}$, *the opening and closing of a function* f *by a function* g *are given respectively, by,* $f \circ g = (f \ominus g) \oplus g$ *and* $f \bullet g = (f \oplus g) \ominus g$.
Corollary 3. *The shadow of the opening and closing of* f *by* g, *where* $f, g \in \mathcal{F}$, *is given correspondingly by,* $\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ \mathcal{S}(g)$ *y* $\mathcal{S}(f \bullet g) = \mathcal{S}(f) \bullet \mathcal{S}(g)$.

Proof: As a consequence of **Corollaries 1** and **2** it is not difficult to see that,
$\mathcal{S}(f \circ g) = \mathcal{S}((f \ominus g) \oplus g) = \mathcal{S}\big((f \ominus g)\big) \oplus \mathcal{S}(g) = \big(\mathcal{S}(f) \ominus \mathcal{S}(g)\big) \oplus \mathcal{S}(g) = \mathcal{S}(f) \circ \mathcal{S}(g)$ and
$\mathcal{S}(f \bullet g) = \mathcal{S}\big((f \oplus g) \ominus g\big) = \mathcal{S}\big((f \oplus g)\big) \ominus \mathcal{S}(g) = (\mathcal{S}(f) \oplus \mathcal{S}(g)) \ominus \mathcal{S}(g) = \mathcal{S}(f) \bullet \mathcal{S}(g)$.

Theorem 7. *The morphological operations of opening and closing of two functions* $f, g \in \mathcal{F}$ *are found in terms of their shadows and the top function, i. e.,* $f \circ g = \mathcal{T}(\mathcal{S}(f) \circ \mathcal{S}(g))$ *and* $f \bullet g = \mathcal{T}(\mathcal{S}(f) \bullet \mathcal{S}(g))$.

Proof: Applying the top function to the result obtained in **Corollary 3** for both opening and closing in terms of their corresponding shadows, it follows that,
$f \circ g = \mathcal{T}(\mathcal{S}(f \circ g)) = \mathcal{T}(\mathcal{S}(f) \circ \mathcal{S}(g))$ and $f \bullet g = \mathcal{T}(\mathcal{S}(f \bullet g)) = \mathcal{T}(\mathcal{S}(f) \bullet \mathcal{S}(g))$.

The algebraic properties of the mathematical morphology operations with functions are similar to the properties of the mathematical morphology operations with sets. Except for terminology and notation, the proof of these properties are the same and can be based on the equalities established between shadow sets according to **Corollaries 1**, **2** and **3**. We remark the important fact that all equalities established involving the shadow of a morphological operation between functions and the corresponding shadows using the same operation with sets are the *algebraic mechanism* that associates the mathematical morphological operations on functions with the same operations acting on sets. Also, the domain of the erosion and dilation functions is expressed by the same morphological operation between the domain sets of the respective functions. By analogy of nomenclature, function g is called a *structuring function* that interacts with f. The mathematical expressions given in **Theorems 5** y **6** are of capital importance for applications in digital and image processing as well as in the mathematical analysis and numerical evaluation of erosion and dilation as will be detailed next by implementing their corresponding computer algorithms using any programming language [19,20].

V. ANALYIS AND NUMERICAL EVALUATION

In this section, which is the central part of our paper, we first exhibit an example of the analysis required to find the algebraic expressions for the elementary morphological operations of erosion and dilation between two simple functions f and g. We include in this example the correponding graphs of the algebraic results. Secondly we give simple algorithms written with the programming constructs provided by PTC's (Parametric Technology Corporation) *Mathcad Prime* software working environment [24]. Additional functional graph examples are provided to illustrate the corresponding numerical evaluation.

An Example of Algebraic Determination of Erosion and Dilation

Consider the following functions, $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g: [-\varepsilon, \varepsilon] \to \mathbb{R}$ such that $g(x) = -x^2 + 1$ (see Fig. 1). Our goal is to find the erosion and dilation of these functions by means of the mathematical formulae given in **Theorems 5** y **6**. For the erosion of f by g, let $h_x(a) = f(x + a) - g(a)$ where $a \in [-\varepsilon, \varepsilon]$. After substitution of the given functions, doing some simple calculations and simplifying, we have that $h_x(a) = 2a^2 + 2ax + x^2 - 1$. Therefore, $h_x'(a) = 4a + 2x$ and $h_x''(a) = 4 > 0 \quad \forall a \in (-\varepsilon, \varepsilon)$. Thus, $h_x'(a) = 0 \Leftrightarrow a = -\frac{x}{2} \ \forall x \in (-2\varepsilon, 2\varepsilon)$. Since $h_x'' > 0$ and $a = -\frac{x}{2}$ is the only relative extremum in $[-\varepsilon, \varepsilon]$, then $a = -\frac{x}{2}$ is the absolute minimum in $[-\varepsilon, \varepsilon]$. Hence, $\forall x \in (-2\varepsilon, 2\varepsilon)$, h_x' has an absolute minimum at $a = -\frac{x}{2}$. On the other hand, if $x > 2\varepsilon$ then $h_x' > 0$, so that h is strictly increasing and its absolute minimum occurs at $a = -\varepsilon$ and $h_x(-\varepsilon) = x^2 - 2\varepsilon x + 2\varepsilon^2 - 1$. However, if $x < -2\varepsilon$ then $h_x' < 0$ and the funtion h_x is strictly decreasing and attains its minimum at $a = \varepsilon$ where $h_x(\varepsilon) = x^2 + 2\varepsilon x + 2\varepsilon^2 - 1$. In consequence,

$$(f \ominus g)(x) = \begin{cases} x^2 + 2\varepsilon x + 2\varepsilon^2 - 1 & \text{if} \quad x < -2\varepsilon, \\ \dfrac{x^2}{2} - 1 & \text{if} \quad -2\varepsilon \le x \le 2\varepsilon, \\ x^2 - 2\varepsilon x + 2\varepsilon^2 - 1 & \text{if} \quad x > 2\varepsilon. \end{cases}$$

For the dilatation of f by g, let $h_x(a) = f(x - a) + g(a)$ where $a \in [-\varepsilon, \varepsilon]$, then $h_x(a) = x^2 - 2ax + 1$. Thus, if $x \ge 0$, function h_x is decreasing and reaches its maximum at $a = -\varepsilon$ and if $x < 0$, then h_x is increasing and its absolute maximum happens at $a = \varepsilon$. Therefore,

$$(f \oplus g)(x) = \begin{cases} x^2 - 2\varepsilon x + 1 & \text{if} \quad x < 0, \\ x^2 + 2\varepsilon x + 1 & \text{if} \quad x \ge 0. \end{cases}$$

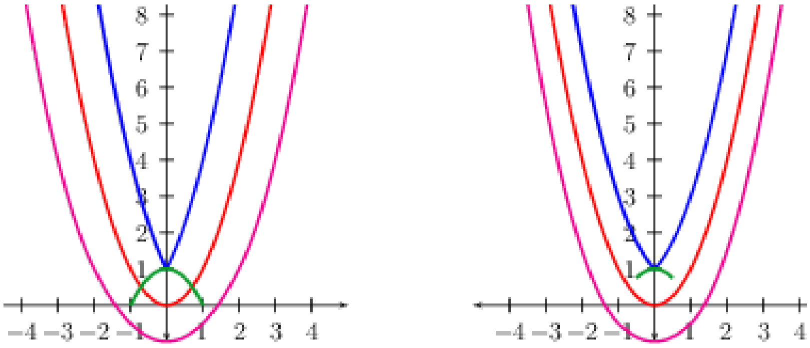

Figure 1: Graphs of the functions: source function f (red), structuring function g (green), erosion function $f \ominus g$ (magenta) and dilation function $f \oplus g$ (blue); function f has the same domain in both cases although the domain of function g is different, in the left graph $\varepsilon = 1$ and in the right graph $\varepsilon = 1/2$.

For the selected structuring function g we can see in the graphs of Fig. 1, that the morphological erosion diminishes the values of f and the morpholical dilation augments its values. Hence, $f \ominus g$ and $f \oplus g$, are separated from f below and above respectively. The numerical change in the values of f depends on the semilength ε of the interval domain of g as well as its parabolic shape.

Corollary 4. *If $g = 0$ and if $B = D_g$, then the definitions for morphological erosion and dilatation between functions are simplified, respectively, to the following expressions:*

$$(f \ominus B)(x) = \bigwedge_{b \in B} f(x+b) = \bigwedge_{b \in B} f_{-b} \;\; with \;\; (x+b) \in D_f \;\; and$$

$$(f \oplus B)(x) = \bigvee_{b \in B} f(x-b) = \bigvee_{b \in B} f_b \;\; with \;\; (x-b) \in D_f.$$

In the simplified formulae, the functional value of the erosion and dilation at point x is the minimum or maximum, respectively, taking into account that the origin of the window set defined by the structuring element B (support domain of g) is translated to x. Also, $D_{f \ominus B} = D_f \ominus B$ and $D_{f \oplus B} = D_f \oplus B$. Since $g = 0$ the domain set $D_g = B$ is called a *flat structuring element* and in this case we speak of "flat mathematical morphology" or "plane mathematical morphology".

Examples of Numerical Evaluation of Erosion and Dilation

The software used to do the numerical evaluation of the morphological operations with functions including the resulting graphs is PTC's Mathcad Prime which is a professional computational environment for scientists, engineers, and technicians. Besides its numerical, functional, and graphical capabilities it also has built-in programming constructs to design and test algorithms. The specific implementation of the algebraic results obtained in our previous example using Mathcad consists of the following steps:

1) Source and structuring functions definition. From our example, $f(x) \coloneqq x^2$ and $g(x,\varepsilon) \coloneqq \mathbf{if}(|x| < \varepsilon, -x^2 + 1, 0)$, where **if**(*condition*, *then clause*, *else clause*) is a programming built-in construct. When condition evaluates to true the then clause is executed, otherwise the else clause is considered. Mathcad uses $\coloneqq$ as the definition operator.

2) Establish the domain corresponding to f in such a way that includes the domain of g for different values of the ε parameter since $D_g = [-\varepsilon, \varepsilon]$. For our example, we selected $D_f = [-4.25, 4.25]$ with a step size of 0.001 written implicitly in Mathcad by specifying the domain interval as $x \coloneqq -4.25, -4.249..4.24$. Mathcad uses two low dots to specify a numerical range where the second comma separated value establishes the desired step size for numerical evaluation along the given range.
3) Code the the erosion and dilation functions, $f \ominus g$ and $f \oplus g$, as parametric functions, i. e., $\mathrm{ero}(x,\varepsilon) \coloneqq \mathbf{if}(x < -2 \cdot \varepsilon, x^2 + 2 \cdot \varepsilon \cdot x + 2 \cdot \varepsilon^2 - 1, \mathbf{if}(x > 2 \cdot \varepsilon, x^2 - 2 \cdot \varepsilon \cdot x + 2 \cdot \varepsilon^2 - 1, 0.5 \cdot x^2 - 1))$ and $\mathrm{dil}(x,\varepsilon) \coloneqq \mathbf{if}(x < 0, x^2 - 2 \cdot \varepsilon \cdot x + 1, x^2 + 2 \cdot \varepsilon \cdot x + 1)$.

4) Based on Mathcad's graphical capabilities, trace functions $f, g, \mathrm{ero}, \mathrm{dil}$ as 2D plots considering the specified x numerical domain for $\varepsilon = 0.5,1,1.5$. Fig. 2 displays the corresponding graphs. Observe that the erosion and dilation functions change their values as the parameter ε changes.

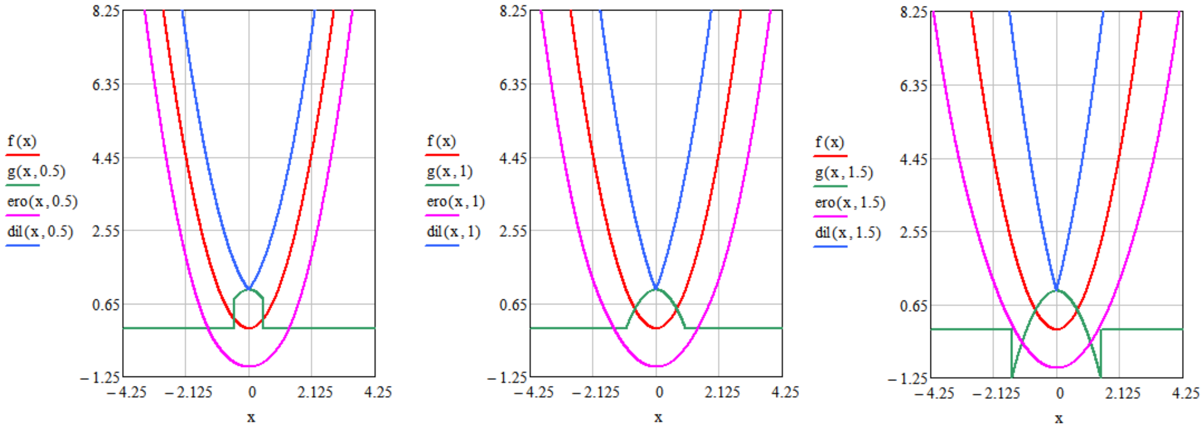

Figure 2: Graphs of the functions: source function f (red), structuring function g (green), eroded function ero (magenta) and dilated function dil (blue); function f has the same domain in all cases but the domain of function g is different, specifically, $\varepsilon = 0.5$ (left graph), $\varepsilon = 1$ (middle graph), and $\varepsilon = 1.5$ (right graph).

In the following example we find numerically the erosion and dilation of a positive semicircle function of radius ϱ with a structuring function that also is a positive semicircle of radius ε where $\varepsilon < \varrho$, meaning that the structuring semicircle is smaller then the source semicircle. Thus, for computational purposes, the source and structuring functions are defined in Mathcad as $f(x,\varrho) := \mathbf{if}(|x| < \varrho, \sqrt{\varrho^2 - x^2}, 0)$ and $g(x,\varepsilon) := \mathbf{if}(|x| < \varepsilon, \sqrt{\varepsilon^2 - x^2}, 0)$. Fig. 3 shows the graphs of $f(x,\varrho)$ and $g(x,\varepsilon)$ where $\varrho = 4$ is fixed and $\varepsilon = 0.5,1,1.5$.

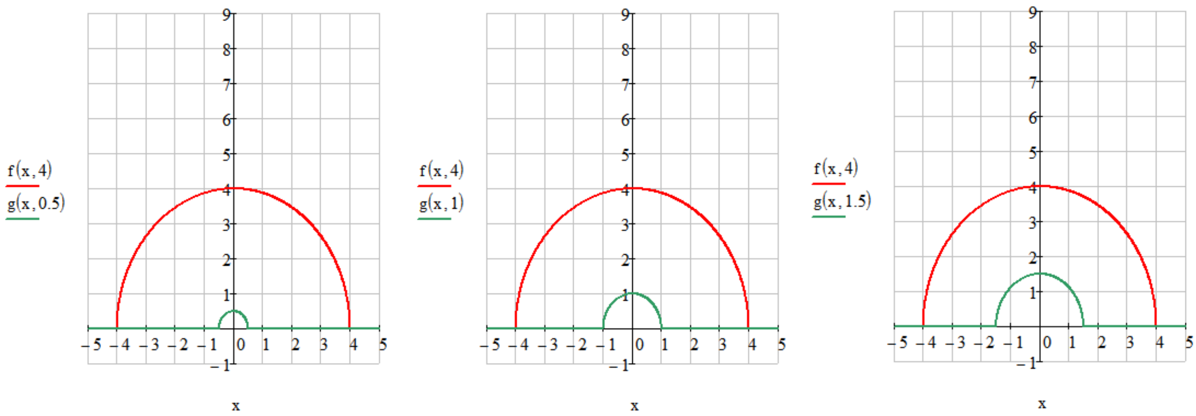

Figure 3: Graphs of the source and structuring semicircle functions, $f(x,\varrho)$ (red) and $g(x,\varepsilon)$ (green). For the source semicircle the radius $\varrho = 4$ whereas for the smaller semicircle the radius takes values $\varepsilon = 0.5,1,1.5$.

To find numerically the erosion $(f_\varrho \ominus g_\varepsilon)(x) = f(x,\varrho) \ominus g(x,\varepsilon)$ and the dilation $(f_\varrho \oplus g_\varepsilon)(x) = f(x,\varrho) \oplus g(x,\varepsilon)$ the simple Mathcad code shown below is used to perform the calculations:

$$\mathrm{ero}(x,h,\varrho,\varepsilon) := \left| \begin{array}{l} \mu \leftarrow \infty \\ \mathbf{for}\ a \in -\varepsilon, -\varepsilon + h \,..\, \varepsilon \\ \mu \leftarrow \min(\mu, f(x+a,\varrho) - g(a,\varepsilon)) \end{array} \right.$$

$$\mathrm{dil}(x,h,\varrho,\varepsilon) := \left| \begin{array}{l} \mu \leftarrow -\infty \\ \mathbf{for}\ a \in -\varepsilon, -\varepsilon + h \,..\, \varepsilon \\ \mu \leftarrow \max(\mu, f(x-a,\varrho) + g(a,\varepsilon)) \end{array} \right.$$

The top to bottom programming instructions in each Mathcad procedure show two steps. The first step initializes vector μ with the maximum or minimum machine number. In Mathcad, $\infty = 10^{307}$ and $-\infty = -10^{307}$. The second step is a finite **for** loop that increases the value of $a \in [-\varepsilon, \varepsilon]$ by the predefined step size $h = 0.001$ as selected for this example. Once the **for** loop ends, vector μ contains the function values respectively of erosion or dilation. Fig. 4 displays the results obtained for the parameters ϱ and ε as given in Fig. 3. To simplify parameter passing in the Mathcad procedures ero and dil, the values of the step size h and of the source semicircle radius ϱ are declared as constants which in Mathcad are specified as $h := 0.001, \varrho := 4$, before invoking either procedure.

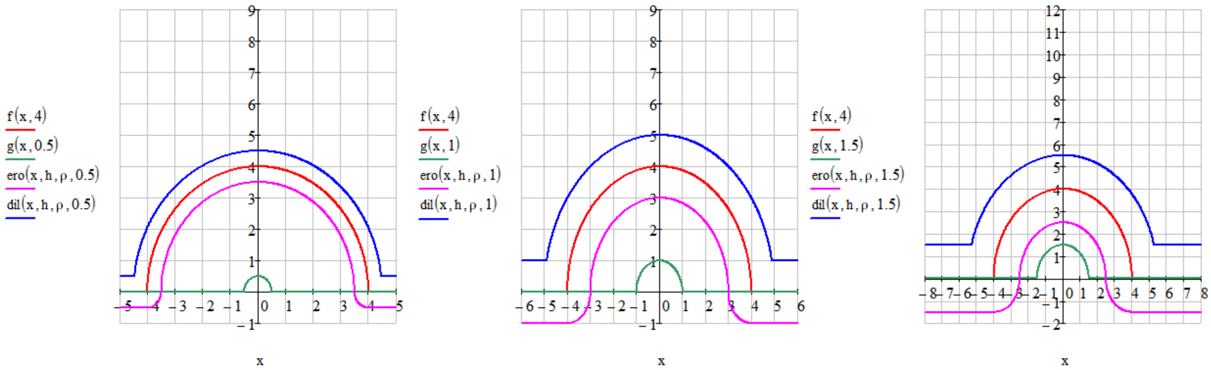

Figure 4: Graphs of the erosion numerical function (magenta) and dilation numerical function (blue) between the source semicircle function $f(x, \varrho)$ (red) by the structuring semicircle function $g(x, \varepsilon)$ (green). The source semicircle radius is 4 whereas for the structuring semicircle the radius takes values 0.5, 1. And 1.5.

Our last example illustrates the case of a flat structuring function for a numerical application of **Corollary 4**. In particular, in the context of Mathcad, the chosen source function is periodic defined by $f(x) := \cos(\pi x/2) + 1$, the flat structuring function is defined as $g(x, \varepsilon) := \mathbf{if}(|x| \leq \varepsilon, 0, -0.5)$, and the domain variable runs along the numerical range specified as $x := -6, -5.99 \,..\, 6$. Note that $g = 0$ in the interval $[-\varepsilon, \varepsilon]$ and that the assigned value of -0.5 outside it is for graphing purposes only since it is not considered during computation of the morphological erosion or dilation. Fig. 5 display the graphs of both functions where the support of function g, i. e., D_g varies in length by changing the value of ε.

To find numerically the erosion $(f \ominus g_\varepsilon)(x) = f(x) \ominus g(x, \varepsilon)$ and dilation $(f \oplus g_\varepsilon)(x) = f(x) \oplus g(x, \varepsilon)$, the Mathcad code given below, in which g_ε does not appear (since it is zero), is used to perform the calculations:

$$\text{ero1}(x, h, \varepsilon) := \left| \begin{array}{l} \quad \mu \leftarrow \infty \\ \mathbf{for}\ a \in -\varepsilon, -\varepsilon + h \,..\, \varepsilon \\ \mu \leftarrow \min(\mu, f(x + a)) \end{array} \right.$$

$$\text{dil1}(x, h, \varepsilon) := \left| \begin{array}{l} \quad \mu \leftarrow -\infty \\ \mathbf{for}\ a \in -\varepsilon, -\varepsilon + h \,..\, \varepsilon \\ \mu \leftarrow \max(\mu, f(x - a)) \end{array} \right.$$

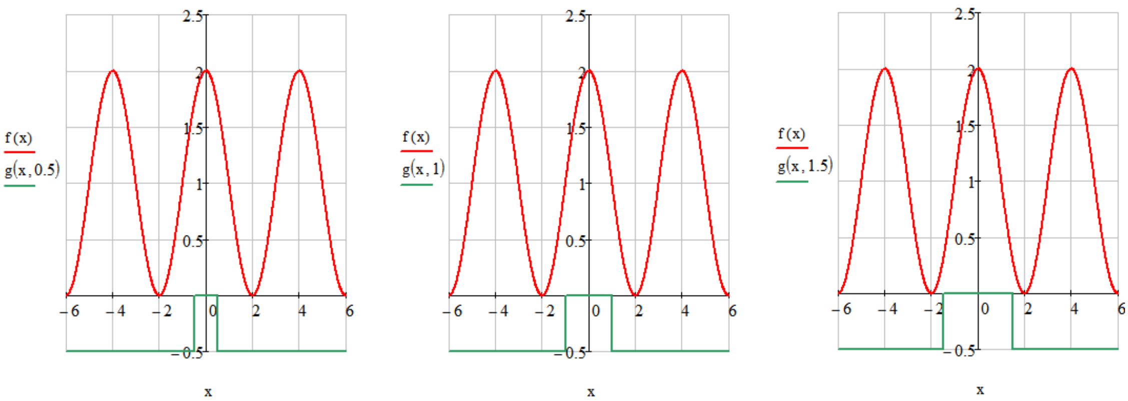

Figure 5: Graphs of the source and flat structuring functions, $f(x)$ (red) and $g(x, \varepsilon)$ (green). Although f is a periodic function only a portion of it is shown; the length values where $g = 0$ are given by $2\varepsilon = 1,2,3$.

In similar fashion to the previous explanation, the **for** loop increases the value of $a \in [-\varepsilon, \varepsilon]$ by the predefined step size $h = 0.01$ as selected for this last example. When the **for** cycle ends, vector μ contains the function values respectively of erosion or dilation. Fig. 6 provides the results obtained for the parameter ε as given in Fig. 5. Again, to simplify parameter passing in the Mathcad procedures ero1 and dil1, the value of the step size h is declared as the constant specified as $h := 0.01$ before calling either procedure.

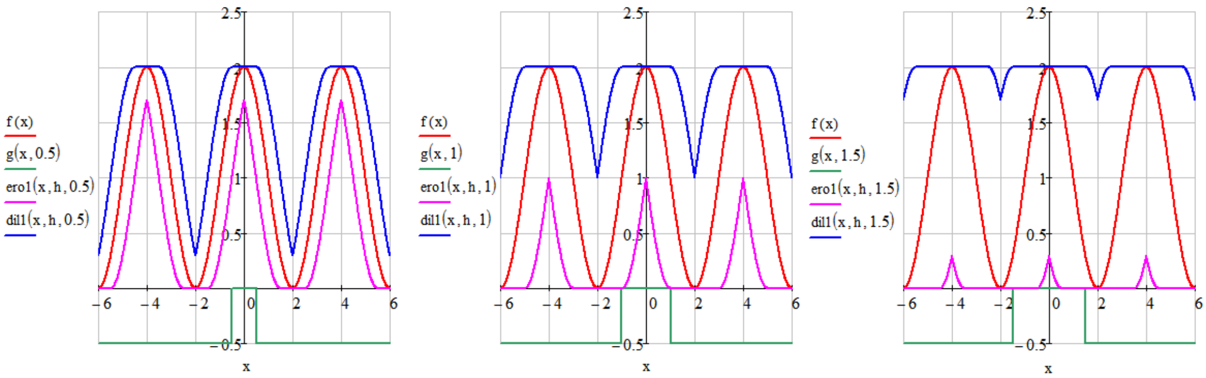

Figure 6: Graphs of the erosion numerical function (magenta) and dilation numerical function (blue) between the source periodic function $f(x)$ (red) by the flat structuring function $g(x, \varepsilon) = 0$ (green). The greater the value of ε extending the domain of g, the more pronounced is the "flattening" of the corresponding eroded and dilated function envelopes.

VI. CONCLUSION

In this paper we have exposed with sufficient detail the conceptual background and the basic theory of mathematical morphology, giving a detailed example of algebraic determination and additional ones for the numerical evaluation using PTC's Mathcad Prime software in the case of morphological erosion and dilation with functions. Also, we have consider a step by step development in several proofs on the basic morphological operations using functions (erosion, dilation, opening, and closing). The ideas of the top function and the shadow set are key concepts to build a bridge connecting function morphology with set morphology including the case of flat structuring elements which are the most employed in signal and image grayscale processing. To get more knowledge in this scientific field within applied mathematics we suggest to the interested reader to consult the classic references [1], [3] and [13], or more recent developments in the technical literature such as [10], [15], [21], [22] and [23].

Acknowledgments

Gonzalo Urcid thanks SNI-CONACYT in Mexico City for grant No.22036.

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