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Direct Integration and Work Principle as New Approach in Bending Analyses of Isotropic Rectangular Plates

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--Abstract--
Following the work of Timoshenko &Woinowsky–Krieger(1959) and subsequent researchers, the traditional approaches in analysis of isotropic rectangular plates has been to assume a differential continuous shape function and to employ strain energy and variational principles. Theseapproaches are grossly limited by the difficulty in assuming suitable shape functions, especially for young structural analysts. A new approach to bending analysis of rectangular plates is presented in this paper. Unlike the traditional approaches that first of all assume a shape function, the shape function in this new approach is obtained by direct integration of the governing differential equation for isotropic rectangular plates. This shape function is in the form of Taylor series in two dimensions.Also, rather than the strain energy and variational principles of the traditional approaches, the plate analysis in this new approach is by the simple principle of equilibrium of works performed by the action (load) and resistance (plate),through which equations for deflection and bending moments are obtained. The maximum deflection at the center of SSSS plateand the maximum bending moment in X-direction obtained by this new approach compare favourably with those fromTimoshenko &Woinowsky–Krieger(1959) for aspect ratios from 1.0 to 2.0 at 0.1 increments. It is therefore, recommended that this new approach could be more easily and better used for the analysis of isotropic rectangular plates.

Key Words: *Rectangular plate; Direct integration; Strain energy; Variational principles; Governing differential equation; Shape function; Equilibrium ofworks; Taylor series*

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I. Introduction

The analysis of isotropic rectangular plates has been a major subject matter over the century. Three approaches to it have been distinguished, namely the equilibrium approach, thevariational approach, and the numerical approach. The equilibrium approach involves the direct integration of the governing partial differential plate equation. Earlier scholars such as Navier(1823) and Levy (1899)assumed some functions in forms of trigonometric series that satisfied the boundary conditions of the particular plates and substituted them into the governing equation before integration.Later scholars employing the equilibrium approach have continued to assume shape functions and to usetrigonometric and hyperbolic-trigonometric series. Probably the greatest problem with this approach is that it is very difficult to assume satisfying shape functions for many plates of various boundary conditions(Hencky, 1913; Wojtaszak, 1937; Timoshenko 1938; Evans, 1939; Young, 1940; Timoshenko &Woinowsky–Krieger, 1959; Iyenger 1988, Hutchinson, 1992; Ye 1994; Ugural 1999; Wang *et al*, 2002; Taylor &Govindjee, 2004; Wang & El-Sheikh, 2005; Imrak and Gerdemeli, 2007; Liu et al., 2007).This difficulty led some scholars to try the use of the variational approach based on energy principle. However, just like the equilibrium approach, this approach also makes use of assumed shape functions, most of which are functions of trigonometric series. The limitations in the use of both equilibrium and variational approaches led to the use of numerical approaches such as finite element and finite difference methods (Ritz, 1908, Fok, 1980; Rao, 1989; Cook, Malkus and Plesha, 1989; Reddy, 2002; Szilard, 2004; Ibearugbulem et al., 2011).

This paper presents a new approach in bending analysis of isotropic rectangular plates using direct integration and work principle. Unlike the traditional equilibrium and variational approaches that first of all assume shape functions, this new approach carries out direct integration of the governing differential equation of isotropic rectangular plate to obtain suitable shape functions in the form of Taylor series in two dimensions. Also, rather than the strain energy and variational principles of the traditional approaches, the plate analysis in this new approach is by the simple principle of equilibrium of works performed by the action (load) and resistance (plate), through which equations for deflection and bending moments were obtained.

II. Governing Equation Of Rectangular Plate

The Governing differential equation for isotropic rectangular plate under combined action of lateral (transverse) and in-plane loads as given by Ventsel and Krauthammer (2001) is as stated in equation 1.

$$\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \frac{\partial^4 W}{\partial Y^4} = \frac{1}{\mathrm{D}}\left(\mathrm{q} + \mathrm{Nx}\frac{\partial^2 W}{\partial X^2} + 2Nxy\frac{\partial^2 W}{\partial X \partial Y} + Ny\frac{\partial^2 W}{\partial Y^2}\right) \quad (1)$$

In this equation q is thelateral load; Nx and Ny are in-plane loads in x, y directions respectively;Nxy is in-plane shearing load; W is the shape function; and D is the plates flexural rigidity.

For a plate undergoing free lateral vibration, the governing equation as given by Chakraverty(2009) is as shown in equation 2.

$$\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \frac{\partial^4 W}{\partial Y^4} = \frac{\rho h}{\mathrm{D}}\frac{\partial^2 \mathrm{W}}{\partial t^2} \quad (2)$$

According to Szilard(2004),

$$\frac{\partial^2 \mathrm{W}}{\partial t^2} = \omega^2 W \quad (3)$$

By the principle of superposition, for combined lateral and in-plane loaded plate undergoing lateral vibration,equations (1), (2), and (3) can be jointly written as shown in equation 4.

$$\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \frac{\partial^4 W}{\partial Y^4} = \frac{1}{\mathrm{D}}\left(\mathrm{q} + \mathrm{Nx}\frac{\partial^2 W}{\partial X^2} + 2Nxy\frac{\partial^2 W}{\partial X \partial Y} + Ny\frac{\partial^2 W}{\partial Y^2} + \rho h\omega^2 W\right) \quad (4)$$

In this equation $\rho h\omega^2 W$istheinertiaforceduetovibration.

Equation (4) can simply be stated as in equation 5:

$$\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \frac{\partial^4 W}{\partial Y^4} = \frac{\mathrm{FF}}{\mathrm{D}} \quad (5)$$

Where FF is the applied load on the plategiven specified in equation 6.

$$FF = \mathrm{q} + \mathrm{Nx}\frac{\partial^2 W}{\partial X^2} + 2Nxy\frac{\partial^2 W}{\partial X \partial Y} + Ny\frac{\partial^2 W}{\partial Y^2} + \rho h\omega^2 W \quad (6)$$

Equation (5) can be interpreted as action and reaction being equal in magnitude. The LHS of the equation represents the magnitude of resistance force offered by the plate as it changes from itsoriginally straight configuration to a bent form under the effect of the load acting on it, which is represented by the RHS.

III. Work Principle

Work is defined mathematically as the product of force and distance traveled by the force. Let the three forces FF1, FF2, and FF3 acting on the plate as shown in figure 1amove through distances W4, W5, and W7 respectively as shown in figure 1b, where the plate is assumed to be divided into finite portions as shown in the figure.

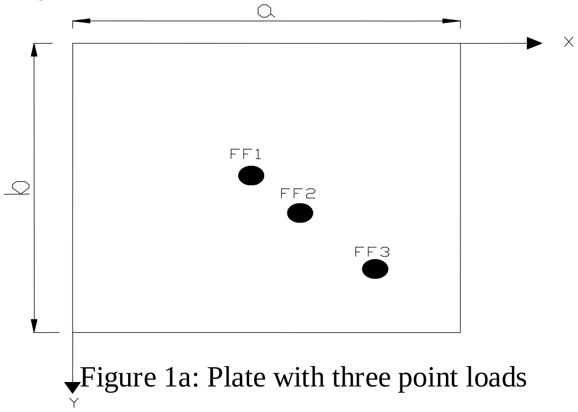

Figure 1a: Plate with three point loads

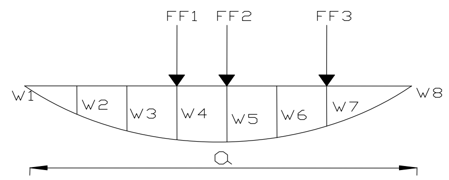

Figure 1b: Bent configuration of the plate along x direction

From equation (5), the force of resistance of the plate is constant for all the portions of plate. Let the total downward work performed by the loads beWkf.Let the total upward work performed by all the portions of the plate be Wkp.This can be expressed as in equation 7.

$$Wkf = \mathrm{FF1} * \mathrm{W4} + \mathrm{FF2} * \mathrm{W5} + \mathrm{FF3} * \mathrm{W7} \quad (7)$$

For finite number of point loads on the plate, equation (7) can be writing mathematically as in equation 8.

$$Wkf = \sum_{\theta=1}^{n} FF\theta * W(x,y)\theta \quad (8)$$

In this equation θ is the number identifying each load, FFθ is the load in question and W(x,y)θ is the magnitude of deflection at the load position.For infinite number of loads such as uniformly distributed load over an area on the plate bounded byx1≤ x ≤ x2 and y1≤ x ≤ y2, equation (7) can be written as in equation 9.

$$Wkf = \int_{x1}^{x2}\int_{y1}^{y2} FF * W(x,y)\,dx\,dy \qquad (9)$$

Let the total upward work performed by all the portions of the plate beWkp. This can be expressed as in equation 10.

$$Wkp = \frac{\nabla^4 W}{D}(W1 + W2 + W3 + W4 + W5 + W6 + W7) \qquad (10)$$

Since, the plate is divided into infinite portions, equation (10) can be represented mathematically as in equation 11.

$$Wkp = \int_0^a\int_0^b \frac{\nabla^4 W}{D} W(x,y)\,dx\,dy \qquad (11)$$

Equation 5 may be expressed as in equation 12.

$$Wkp = Wkf \qquad (12)$$

Substituting equations (8), (9) and (11) into equation (12) results in equation 13.

$$\int_0^a\int_0^b D.[\nabla^4 W(x,y)]W(x,y)\,dx\,dy = \sum_{\theta=1}^{n} FF\theta * W(x,y)\theta + \int_{x1}^{x2}\int_{y1}^{y2} FF * W(x,y)\,dx\,dy \qquad (13)$$

$$\text{Where}\,\nabla^4 W(x,y) = \frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2\partial Y^2} + \frac{\partial^4 W}{\partial Y^4}$$

IV. Solution Of The Governing Equation By Direct Integration

Equation 5 can be written in the form of partial integration as shown in equation 14.

$$\int_0^a\int_0^b \left[\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2\partial Y^2} + \frac{\partial^4 W}{\partial Y^4}\right]dx\,dy = \int\int \frac{FF}{D}\,dx\,dy \qquad (14)$$

Evaluating equation 14 gives:

$$\left[\frac{\partial^3 W}{\partial X^3}y + 2\frac{\partial^2 W}{\partial X\partial Y} + \frac{\partial^3 W}{\partial Y^3}x\right]_{0\,0}^{a\,b} = \frac{FF}{D}(J_4 x + J_3)(K_4 y + K_3)$$

That is

$$\left[\frac{\partial^3 W}{\partial X^3}b + 2\frac{\partial^2 W}{\partial X\partial Y} + \frac{\partial^3 W}{\partial Y^3}a\right] = \frac{FF}{D}(J_4 x + J_3)(K_4 y + K_3) \qquad (15)$$

In a similar way, integrating equation (15) three times gives:

$$b^4 W + 2a^2 b^2 W + a^4 W = FF\left(\frac{J_4 x^4}{24} + \frac{J_3 x^3}{6} + \frac{J_2 x^2}{2} + J_1 x + J_0\right)\left(\frac{K_4 x^4}{24} + \frac{K_3 x^3}{6} + \frac{K_2 x^2}{2} + K_1 x + K_0\right)$$

That is,

$$W = \frac{FF}{D[b^4 + 2a^2 b^2 + a^4]}\left(\frac{J_4 x^4}{24} + \frac{J_3 x^3}{6} + \frac{J_2 x^2}{2} + J_1 x + J_0\right)\left(\frac{K_4 x^4}{24} + \frac{K_3 x^3}{6} + \frac{K_2 x^2}{2} + K_1 x + K_0\right)$$

Let $\frac{q}{D[b^4 + 2a^2b^2 + a^4]} = \Omega =$ constant, hence the shape function (solution) becomes:

$$W = \Omega\left(\frac{J_4 x^4}{24} + \frac{J_3 x^3}{6} + \frac{J_2 x^2}{2} + J_1 x + J_0\right)\left(\frac{K_4 x^4}{24} + \frac{K_3 x^3}{6} + \frac{K_2 x^2}{2} + K_1 x + K_0\right) \qquad (16)$$

Equation 16 can be written in a short form as in equation 17:

$$W = \Omega\sum_{m=0}^{4}\sum_{n=0}^{4}\frac{J_m}{m!}x^m\frac{K_n}{n!}y^n \qquad (17)$$

For a rectangular plate whose lengths in x and y directions are a and b, non-dimensional parameters in x and y directions are defined as in equations 18 and 19:

$$R = \frac{x}{a}. \quad That\ is \quad x = aR \qquad (18)$$

$$Q = \frac{y}{b}. \quad That\ is \quad y = bQ \qquad (19)$$

Let the aspect ratio be $\gamma = \frac{a}{b}$ or $\lambda = \frac{b}{a}$, where $b \geq a$.

Substituting equations (6) and (7) into equation (5) gives equation 20.

$$W = \Omega \sum_{m=0}^{4} \sum_{n=0}^{4} \alpha_m R^m . \beta_n Q^n \qquad (20)$$

$$Where\ \alpha_m = \frac{J_m . a^m}{m!} = constant \qquad (21)$$

$$\beta_n = \frac{K_n . b^n}{n!} = constant \qquad (22)$$

Substituting equations (18) and (19) into expanded equation (14) and rearranging gives equation 23.

$$\int_0^1 \int_0^1 \left[\frac{\partial^4 W}{\gamma^4 \partial R^4} W + 2 \frac{\partial^4 W}{\gamma^2 \partial R^2 \partial Q^2} W + \frac{\partial^4 W}{\partial Q^4} W \right] dR\, dQ = \frac{b^2}{\gamma} \sum_{\theta=1}^{n} FF\theta * W\theta + \frac{b^4}{D} \int_{R1}^{R2} \int_{Q1}^{Q2} FF.W\, dR\, dQ \qquad (23)$$

V. Application Of The Shape Function In The Work Equation For Ssss Plate Under Lateral Load

The boundary conditions for SSSS plate are as shown in equations 24-27:

$$w(R = 0) = 0;\ w(R = 1) = 0 \qquad (24)$$

$$\frac{\partial^2 w(R = 0)}{\partial R^2} = w^{IIR}(R = 0) = 0;\ \frac{\partial^2 w(R = 1)}{\partial R^2} = w^{IIR}(R = 1) = 0 \qquad (25)$$

$$w(Q = 0) = 0;\ w(Q = 1) = 0 \qquad (26)$$

$$\frac{\partial^2 w(Q = 0)}{\partial Q^2} = w^{IIQ}(Q = 1) = 0;\ \frac{\partial^2 w(Q = 0)}{\partial Q^2} = w^{IIQ}(Q = 1) = 0 \qquad (27)$$

Substituting equations 24-27 into equation (20) and solving the arising simultaneous equations give:

$\alpha_0 = \beta_0 = \alpha_2 = \beta_2 = 0;\ \alpha_1 = \alpha_4;\ \alpha_3 = -2\alpha_4;\ \beta_1 = \beta_4;\ \beta_3 = -2\beta_4$

Substituting these coefficients into equation (20) gives the particular shape function, w(R,Q) for SSSS plate as:

$w(R, Q) = \Omega(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)\alpha_4 \beta_4$. Let $A = \Omega . \alpha_4 \beta_4$, then

$$w(R, Q) = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \qquad (28)$$

The fourth derivatives of w(R,Q) are as shown in equations 29-31.

$$\frac{\partial^4 w(R, Q)}{\partial R^4} = 24A(Q - 2Q^3 + Q^4) \qquad (29)$$

$$\frac{\partial^4 w(R, Q)}{\partial Q^4} = 24A(R - 2R^3 + R^4) \qquad (30)$$

$$\frac{\partial^4 w(R, Q)}{\partial R^2 \partial Q^2} = 144A(R - R^2)(Q - Q^2) \qquad (31)$$

The products of these derivatives and w(R,Q) are as given in equations 32-34.

$$\frac{\partial^4 w(R, Q)}{\partial R^4} W = 24A^2(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)^2 = 24A^2(R - 2R^3 + R^4)(Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8) \qquad (32)$$

$$\frac{\partial^4 w(R, Q)}{\partial Q^4} W = 24A^2(R - 2R^3 + R^4)^2(Q - 2Q^3 + Q^4) = 24A^2(R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)(Q - 2Q^3 + Q^4) (33)$$

$$\frac{\partial^4 w(R, Q)}{\partial R^2 \partial Q^2} W = 144A^2(R - R^2)(Q - Q^2)(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) = 144A^2(R^2 - 2R^4 + R^5 - R^3 + 2R^5 - R^6)(Q^2 - 2Q^4 + Q^5 - Q^3 + 2Q^5 - Q^6) \qquad (34)$$

Integrating these products within the closed domains of the entire area of the plate in x and y axes gives equations 35-37.

$$\int_0^1\int_0^1 \frac{\partial^4 W}{\gamma^4 \partial R^4} W\,dR\,dQ = 24A^2(0.2)(0.049206) = \frac{0.23619A^2}{\gamma^4} \quad (35)$$

$$\int_0^1\int_0^1 2\frac{\partial^4 W}{\gamma^2 \partial R^2 \partial Q^2} W\,dR\,dQ = 2*[144A^2(0.040476)(0.040476)] = \frac{0.471837A^2}{\gamma^2} \quad (36)$$

$$\int_0^1\int_0^1 \frac{\partial^4 W}{\partial Q^4} W\,dR\,dQ = 24A^2(0.049206)(0.2) = 0.23619A^2 \quad (37)$$

Substituting equations (35), (36), and (37) into equation (11) gives the work of resistance of the rectangular plate as:

$$Wkp = \frac{D}{b^4}\left(\frac{0.23619A^2}{\gamma^4} + \frac{0.471837A^2}{\gamma^2} + 0.23619A^2\right)ab$$

$$= \frac{D.A^2}{b^2}\left(\frac{0.23619}{\gamma^3} + \frac{0.471837}{\gamma} + 0.23619\gamma\right) \quad (38)$$

Integrating the shape function within the closed domains of small area of the plate in x and y axes gives equation 39.

$$\int_{R1}^{R2}\int_{Q1}^{Q2} W\,dR\,dQ = A\left(\frac{R^2}{2} - 2\frac{R^4}{4} + \frac{R^5}{5}\right)_{R1}^{R2}\left(\frac{Q^2}{2} - 2\frac{Q^4}{4} + \frac{Q^5}{5}\right)_{Q1}^{Q2} \quad (39)$$

Substituting equation (39) into equation (9) gives the work performed by uniform load distributed over an area within R1 ≤ R ≤ R2 and Q1 ≤ Q ≤ Q2 as shown in equation 40.

$$Wkf = FF*A.\left(\frac{R^2}{2} - 2\frac{R^4}{4} + \frac{R^5}{5}\right)_{R1}^{R2}\left(\frac{Q^2}{2} - 2\frac{Q^4}{4} + \frac{Q^5}{5}\right)_{Q1}^{Q2} \quad (40)$$

VI. SSSS PLATE UNDER UNIFORM LATERAL LOAD OVER ITS ENTIRE AREA AND A POINT LOAD AT ITS CENTER

There are two components of FF in this case,FF1 (uniform distributed load) and FF2 (point load).

$$FF1 = q \text{ and } FF2 = P \quad (41)$$

Hence, the work performed by FF1 is

$$Wkf1 = q*A.\left(\frac{R^2}{2} - 2\frac{R^4}{4} + \frac{R^5}{5}\right)_0^1\left(\frac{Q^2}{2} - 2\frac{Q^4}{4} + \frac{Q^5}{5}\right)_0^1 ab$$

$$= q*A(0.2)(0.2) = 0.04qAab \quad (42)$$

The work performed by FF2 is

$$Wkf2 = P*w\left(\frac{1}{2},\frac{1}{2}\right) = P*A\left[\frac{1}{2} - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right]\left[\frac{1}{2} - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right]$$

$$= P*A(0.3125)(0.3125) = 0.097656PA \quad (43)$$

Substituting equations (37), (41), and (42) into equation (13) gives;

$$\frac{D.A^2}{b^2}\left(\frac{0.23619}{\gamma^3} + \frac{0.471837}{\gamma} + 0.23619\gamma\right) = 0.04qAab + 0.097656PA$$

$$\text{That is } \frac{DA}{b^2}\left(\frac{0.23619}{\gamma^3} + \frac{0.471837}{\gamma} + 0.23619\gamma\right) = 0.04qab + 0.097656P$$

Making A the subject of the equation gives:

$$A = \frac{b^2(0.04qb^2\gamma + 0.097656P)}{D\left(\frac{0.23619}{\gamma^3} + \frac{0.471837}{\gamma} + 0.23619\gamma\right)} = \varpi\left(\frac{b^2}{D}\right) \quad (44)$$

$$A = \frac{a^2\lambda^2(0.04qa^2\lambda + 0.097656P)}{D\left(0.23619\lambda^3 + 0.471837\lambda + \frac{0.23619}{\lambda}\right)} = \psi\left(\frac{a^2}{D}\right) \quad (45)$$

For only uniform load,

$$A = \frac{0.04b^4q}{D\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)} = \varpi^u\left(\frac{b^4}{D}\right) \quad (46)$$

$$A = \frac{0.04qa^4}{D\left(0.23619 + \frac{0.471837}{\lambda^2} + \frac{0.23619}{\lambda^4}\right)} = \psi^u\left(\frac{a^4}{D}\right) \qquad (47)$$

For only central point load,

$$A = \frac{0.097656Pb^2}{D\left(\frac{0.23619}{\gamma^3} + \frac{0.471837}{\gamma} + 0.23619\gamma\right)} = \varpi^p\left(\frac{b^2}{D}\right) \qquad (48)$$

$$A = \frac{0.097656Pa^2}{D\left(0.23619\lambda + \frac{0.471837}{\lambda} + \frac{0.23619}{\lambda^3}\right)} = \psi^p\left(\frac{a^2}{D}\right) \qquad (49)$$

Similarly, for only uniform load, maximum moment occurs at R= Q = 1/2

$$Mx = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{a^2\partial R^2} + \mu\frac{\partial^2 w}{b^2\partial Q^2}\right)$$

$$= \frac{12AD}{b^2}\left[\frac{(R-R^2)(Q-2Q^3+Q^4)}{\gamma^2} + \mu(R-2R^3+R^4)(Q-Q^2)\right] \qquad (50)$$

$$Mx_{max} = \frac{0.9375D}{b^2}\left[\frac{1}{\gamma^2} + \mu\right] * \frac{0.04b^4q}{D\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)}$$

$$= \frac{0.0375b^2q\left(\frac{1}{\gamma^2} + \mu\right)}{\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)} = \beta x^u(qb^2) \qquad (51)$$

$$= \frac{0.0375a^2q(\lambda^2 + \mu)}{\left(0.23619\lambda^2 + 0.471837 + \frac{0.23619}{\lambda^2}\right)} = \beta x'^u(qa^2) \qquad (52)$$

$$My = -D\left(\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\left(\mu\frac{\partial^2 w}{a^2\partial R^2} + \frac{\partial^2 w}{b^2\partial Q^2}\right)$$

$$= \frac{12AD}{b^2}\left[\frac{\mu(R-R^2)(Q-2Q^3+Q^4)}{\gamma^2} + (R-2R^3+R^4)(Q-Q^2)\right] \qquad (53)$$

$$My_{max} = \frac{0.9375D}{b^2}\left[\frac{\mu}{\gamma^2} + 1\right] * \frac{0.04b^4q}{D\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)}$$

$$= \frac{0.0375b^2q\left(\frac{\mu}{\gamma^2} + 1\right)}{\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)} = \beta y^u(qb^2) \qquad (54)$$

$$= \frac{0.0375a^2q(\mu\lambda^2 + 1)}{\left(0.23619\lambda^2 + 0.471837 + \frac{0.23619}{\lambda^2}\right)} = \beta y'^u(qa^2) \qquad (55)$$

For only central point load, maximum moment occurs at R= Q = 1/2

$$Mx_{max} = \frac{0.9375D}{b^2}\left[\frac{1}{\gamma^2} + \mu\right] * \frac{0.0976563Pb^2}{D\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)}$$

$$= \frac{0.09155P\left(\frac{1}{\gamma^2} + \mu\right)}{\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)} = \beta x^p P \qquad (56)$$

$$= \frac{0.09155P(\lambda^2 + \mu)}{(0.23619\lambda^4 + 0.471837\lambda^2 + 0.23619)} = \beta x'^p(P) \qquad (57)$$

$$My_{max} = \frac{0.9375D}{b^2}\left[\frac{\mu}{\gamma^2} + 1\right] * \frac{0.09155Pb^2}{D\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)}$$

$$= \frac{0.09155P\left(\frac{\mu}{\gamma^2} + 1\right)}{\left(\frac{0.23619}{\gamma^4} + \frac{0.471837}{\gamma^2} + 0.23619\right)} = \beta y^u(P) \qquad (58)$$

$$My_{max} = \frac{0.09155P(\mu\lambda^2 + 1)}{(0.23619\lambda^4 + 0.471837\lambda^2 + 0.23619)} = \beta y'^p(P) \qquad (59)$$

VII. Results And Discussion

Values of maximum deflection and moments in x and y directions for aspect ratio □ = a/b due to uniform load over the entire area of plate and central point load are presented in tables 1 and 2. Tables 3 , 4,and 5are for aspect ratio λ = b/a. Values of maximum deflections due to uniform load as presented in table 3 were compared with those from Timoshenko &Woinowsky–Krieger (1959) and it was found that the average percentage difference is only 3.03%. The bending moments were also compared and the average percentage difference for moments in x direction is 6.84%. This value is acceptable considering the fact that a small difference in deflection results to large difference in moment. Thus, the shape function derived in this new approach is a good alternative to the shape functions assumed in the traditional approaches. It is therefore, recommended that this new approach of direct integration and work principle could be more easily used for the analysis of isotropic rectangular plates.

Table 1: Maximum deflection and moments due to uniform load over the entire area of plate when the aspect ratio, □ = a/b is used

Aspect ratio, □ = a/b	$A = □^u(b^4/D)$	wmax = 0.097656A = ω0 (b^4/D)	$Mx_{max} = □\, x^u(qb^2)$	$My_{max} = □\, y^u(qb^2)$
	$□^u$	□	$□\, x^U$	$□\, x^U$
0.1	0.0000166	0.00000162	0.00156113	0.000482501
0.2	0.0002505	0.00002447	0.00594265	0.001996544
0.3	0.0011548	0.00011277	0.01235394	0.004691368
0.4	0.0032229	0.00031473	0.01979036	0.00868661
0.5	0.0067767	0.00066179	0.02731857	0.013976944
0.6	0.0118719	0.00115936	0.03425536	0.02040482
0.7	0.0183248	0.00178952	0.04021399	0.027697516
0.8	0.0258053	0.00252004	0.04505842	0.035532645
0.9	0.0339358	0.00331404	0.04882201	0.043598098
1	0.0423631	0.00413702	0.05163008	0.051630081

Table 2: Maximum deflection and moments due to central point load on the plate when the aspect ratio, □ = a/b is used

Aspect ratio, □ = a/b	$A = □^p(Pb^2/D)$	wmax = 0.097656A = ω0 (Pb^2/D)	$Mx_{max} = □\, x^p(P)$	$My_{max} = □\, y^p(P)$
	$□^p$	□	$□\, x^p$	$□\, x^p$
0.1	0.000404969	0.0000396	0.003811227	0.001177947
0.2	0.003040432	0.0002987	0.014508002	0.00487423
0.3	0.009250567	0.0009177	0.030160077	0.011453194
0.4	0.019124793	0.0019210	0.048314876	0.021206911
0.5	0.031816156	0.0032314	0.06669374	0.034122378
0.6	0.046149981	0.0047174	0.083628761	0.049814966
0.7	0.061079151	0.0062413	0.09817575	0.067618869
0.8	0.075861408	0.0076905	0.110002619	0.086747031
0.9	0.090051585	0.0089899	0.11919081	0.106437489
1	0.103425378	0.0101001	0.126046237	0.126046237

Table 3: Maximum deflection due to uniform load over the entire areaof plate when the aspect ratio, λ = b/a is used

Aspect ratio, λ = b/a	A = □^{u}(qa^4/D)	wmax = 0.097656A = ω0 (qa^4/D)		Percentage difference
		Present Work	Timoshenko	
	□^{u}	ω0		
1	0.0423631	0.0041370	0.00406	1.896923993
1.1	0.0507963	0.0049606	0.00485	2.279665362
1.2	0.0590181	0.0057635	0.00564	2.189248402
1.3	0.0668807	0.0065313	0.00638	2.371445845
1.4	0.0742935	0.0072552	0.00705	2.910676375
1.5	0.0812100	0.0079306	0.00772	2.728552455
1.6	0.0876153	0.0085562	0.0083	3.086307993
1.7	0.0935160	0.0091324	0.00883	3.424617875
1.8	0.0989320	0.0096613	0.00931	3.77339465
1.9	0.1038917	0.0101456	0.00974	4.164719054
2	0.1084272	0.0105886	0.01013	4.526816931

Table 4: Maximum moments due to uniform load over the entire area of plate when the aspect ratio, λ = b/a is used

Aspect ratio, λ = b/a	Mx$_{max}$ = □ x^u(qa^2)		Percentage difference	My$_{max}$ = □ y^u(qa^2	My$_{max}$ = □ y^u(qa^2)	Percentage difference
	Present Work	Timoshenko		Present Work	Timoshenko	
	□ x^u			□ y^u		
1	0.05163008	0.0479	7.79	0.05163	0.0479	7.79
1.1	0.05942853	0.0554	7.27	0.05364	0.0493	8.81
1.2	0.06685647	0.0627	6.63	0.05502	0.0501	9.82
1.3	0.07383091	0.0694	6.38	0.05591	0.0503	11.16
1.4	0.08031086	0.0755	6.37	0.05643	0.0502	12.41
1.5	0.08628563	0.0812	6.26	0.05668	0.0498	13.81
1.6	0.09176509	0.0862	6.46	0.05673	0.0492	15.30
1.7	0.09677202	0.0908	6.58	0.05664	0.0486	16.54
1.8	0.10133659	0.0948	6.90	0.05645	0.0479	17.85
1.9	0.10549248	0.0985	7.10	0.05620	0.0471	19.32
2	0.10927429	0.1017	7.45	0.05591	0.0464	20.49

Table 5: Maximum deflection and moments due to central point load on the plate when the aspect ratio, $\lambda = b/a$ is used

Aspect ratio, $\lambda = b/a$	$A = \square^{p}(Pb^2/D)$	wmax = 0.097656A = $\omega 0\ (Pb^2/D)$	$Mx_{max} = \square\ x^{p}(P)$	$My_{max} = \square\ y^{p}(P)$
	$\square^{p}$	$\square_{\square}$	$\square\ x^{p}$	$\square\ y^{p}$
1	0.103425378	0.010100109	0.126046237	0.126046237
1.1	0.112740086	0.011009746	0.119904828	0.108231974
1.2	0.120072367	0.011725787	0.11334647	0.093282842
1.3	0.125601889	0.012265778	0.106654359	0.080767899
1.4	0.129557191	0.012652037	0.100033456	0.070288995
1.5	0.132177404	0.012907917	0.093623105	0.06149753
1.6	0.133690056	0.013055636	0.087511395	0.054097953
1.7	0.134299908	0.013115192	0.081748361	0.047844573
1.8	0.134184764	0.013103947	0.076356915	0.042535547
1.9	0.133495311	0.013036618	0.07134136	0.038006152
2	0.132357082	0.012925463	0.06669374	0.034122378

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