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# Solving Nonlinear Partial Differential Equation Using Painleve Analysis

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**Abstract**

*We employ the use of Painleve analysis in order to solve nonlinear partial differential equation (NLPDE) that passes the panileve test. Backlund transformation is then readily found by using the Painleve truncation expansion. Finally, based on the obtained Backlund transformation, some explicit exact solution of the nonlinear partial differential equation is obtained.*

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## I. Introduction:

Among the various approaches followed to study the integrability of Nonlinear partial differential equations (NLPDE) Painleve analysis has proved to be one of the most successful and widely applied tools [1, 2]. Ablowitz et.al stated that when all the ordinary differential equations (ODE) obtained by exact similarity transform from a given partial differential equation (PDE) have a Painleve property, then the PDE is 'integrable'. The definition of the Painleve property of the ODE was extended to the case of PDE [3]. Briefly a PDE has the Painleve property when the solutions of the PDE are 'single-valued' about the movable, and the singularity manifold is 'non-characteristics'. To be precise, the singularity manifold is determined by $g(z_1,\ldots,z_n) = 0$, where g is an analytic function of $(z_1,\ldots,z_n)$.

The aim of this paper is to use a non perturbative method to built explicit solutions to (NLPDE). A prerequisite task is to investigate whether the chances of successes is high or low, and this can be achieved without a prior knowledge of the solutions, with a powerful tool called the Painleve test. If the equation under study passes the Painleve test then the equation presumed integrable in some sense, and can try to build the explicit information displaying this integrability if on the contrary the test fails the system is not integrable or even chaotic, but it may still be possible to find the solution[4]. The paper is organize as follows: in the next section we will give a brief on the methodology of the propose topic, and in section 3 we will do the implementation and finally the paper will be concluded in section 4.

**Methodology**: We will consider a PDE with a dependent variable v and the independent variable x (for spaces) and t (for time). In solving the candidate equation, we will first write *u(x,t)* as a Laurent series in the complex plane as in [5]

$$u(x,t) = g^{\alpha}(x,t)\sum_{k=0}^{\infty} u_k(x,t)g^k(x,t) \qquad (1)$$

Where *g(x,t)* is the non-characteristic manifold for the poles and α is the negative integer which gives the degree of the most singular term. Second, by substituting the series in to the equation and requiring that the most singular terms vanished, one can obtain the values for α and $u_0(x,t)$. If the next most singular terms are required to vanish, one will obtain the expressions for $u_1(x,t), u_2(x,t),\ldots$ etc, after that the series will be truncated at the constant level term. The truncated series will define a transformation of the dependent variable, which turns out to be crucial in the process of determining the exact closed- form solutions.

**Implementation:** as we know the basic Painleve test of ordinary differential equations (ODE) consist of the following steps

1. Identify all possible dominance balances i. e all singularities of the form $u \sim u_0(z - z_0)^m$
2. If all exponents $m$ are integers, find the resonances where arbitrary constant can appear.
3. If all resonances are are integers check the resonance condition in each Laurent expansion.

And if no obstruction is found in steps 1-3 for any dominant balances then the paileve test is satisfied.
Now consider the Boussinesq equation of the form

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0 \qquad (2)$$

To find the leading order α let

$$u = g^{a}(x,t)u_0(x,t) \qquad (3)$$

Substitute (3) and it derivatives of the form $u_{tt}, u_{xx}$ and etc in (2) to get

$$[-6u_0^2 - 6u_{0xx}u_0]g^{2a} - [6a\,g_{xx}u_0^2 + 24a\,g_x u_{0x}u_0]g^{2a-1} + [6a\,g_x^2u_0^2 - 12a^2g_x^2u_0^2]g^{2a-2}$$
$$+[u_{0tt} - u_{0xx} - u_{0xxxx}]g^{a}$$
$$+[a\,g_{tt}u_0 + 2a\,g_t u_{ot} - a\,g_{xx}u_0 - 2a\,g_x u_{0x} - 4a\,g_{xxx}u_{0x} - 6a\,g_{xx}u_{0xx} - a\,g_{xxxx}u_0$$
$$-4a\,g_x u_{0xxx}]g^{a-1}$$
$$+[a^2g_t^2u_0 - a\,g_t^2u_0 - a^2g_x^2u_0 + a\,g_x^2u_0 - 12a^2g_xg_{xx}u_{0x} + 12a\,g_xg_{xx}u_{0x} - 4a^2g_xg_{xxx}u_0$$
$$+4a\,g_xg_{xxx}u_0 - 3a^2g_{xx}^2u_0 + 3a\,g_x^2u_0 - 6a^2g_x^2u_{0xx} + 6a\,g_x^2u_{0xx}]g^{a-2}$$
$$+[-6a^3g_x^2g_{xx}u_0 + 18a^2g_x^2g_{xx}u_0 - 12a\,g_x^2g_{xx}u_0 - 4a^3g_x^3u_{0x} + 12a^2g_x^3u_{0x} -$$
$$18a\,g_x^2u_{0x}]g^{a-3} +$$
$$[-a^4g_x^4u_0 + 6a^3g_x^4u_0 - 11a^2g_x^4u_0 + 6a\,g_x^4u_0]g^{a-4} = 0$$

(4)

From equation (4), we can see that the most singular powers of g are $2a - 2$ *and* $a - 4$ therefore, equating these powers we get α=-2.this complete the first step.

Hence, the second step is to compute the resonance number j, by collecting terms of each order,g we obtained the most singular terms as $g^{-6}$ in (4). I e

$$(-60g_x^2u_0^2 - 120g_x^4u_0)g^{-6} \qquad (5)$$

Equation (5) will vanish if

$$u_0 = -2g_x^2 \qquad (6)$$

Next to find $u_1$ we let

$$u = g^{-2}(x,t)[u_0(x,t) + g(x,t)u_1(x,t)] \qquad (7)$$

$$\text{I.e. } u = \frac{u_1}{g} - \frac{2g_x^2}{g^2} \qquad (8)$$

Substituting (8) for u into equation (2) and compute again the coefficient of the most singular term ( $g^{-5}$ in this case), we find

$$120u_1g_x^4 - 240g_x^4g_{xx} \qquad (9)$$

This term will be eliminated using

$$u_1 = 2g_{xx} \qquad (10)$$

Next we calculate $u_2$ from which

$$u = g^{-2}(x,t)[u_0(x,t) + g(x,t)u_1(x,t) + g^2(x,t)u_2(x,t) \qquad (11)$$

$$u = u_2 + \frac{2g_{xx}}{g} - \frac{2g_x^2}{g^2} \qquad (12)$$

Equation (12) is then substituted in to the equation (1). It turns out that $u_2$ has to satisfy

$$(u_2)_{tt} - (u_2)_{xx} - 3(u_2^2)_{xx} - (u_2)_{xxxx} = 0 \quad (13)$$

Setting different power terms in *g(x,t)* equal to zero will allow the findings of $u_0, u_1, u_2$ and etc. the series will always be truncated at the constant level term of g, and the coefficient of the constant level will be set to zero. Setting $u_2$ =0, we obtain

$$u = \frac{2g_{xx}}{g} - \frac{2g_x^2}{g^2} = 2\frac{\partial^2}{\partial x^2}\ln g \quad (14)$$

Now to solve equation (2) substitute (14) in the (2), and integrate twice with respect to x to get

$$2g^{-2}[g_x^2 - g_t^2 - 6g_{xx}^2 + 4g_x g_{xxx} + 3g_{xx}^2] + 2g^{-1}[g_{tt} - g_{xx} - g_{xxxx}] \quad (15)$$

Then we take the individual coefficient of different powers of *g*, set them equal to zero, and solve each equation separately;

$$g_{tt} - g_{xx} - g_{xxxx} = 0 \quad (16)$$

$$g_x^2 - g_t^2 - 6g_{xx}^2 + 4g_x g_{xxx} + 3g_{xx}^2 = 0 \quad (17)$$

We notice that equation (16) is linear while (17) is nonlinear

Now to obtain a solitary wave solution, let assume the following form for g

$$g = 1 + ce^{kx + wt + d} \quad (18)$$

Where $c, k, w, and\ d$ are constants.

Substitute (18) in to (16) gives the dispersion relation law

$$w^2 = k^2 + k^4 \quad (19)$$

Note that equation (17) also satisfies the dispersion relations (19).

Using equation (13) and (18) the solution of equation of (2) will now be written as

$$u = \frac{2ck^2 e^{kx + \sqrt{k^2 + k^4}t + d}}{(1 + ce^{kx + \sqrt{k^2 + k^4}t + d})^2}$$

For *c=1* the solution can be written in the form

$$u = \frac{k^2}{2}[1 - \tanh^2\left(\frac{kx + \sqrt{k^2 + k^4}t + d}{2}\right)] = \frac{k^2}{2}\sec h^2\left(\frac{kx + \sqrt{k^2 + k^4}t + d}{2}\right)$$

## Conclusion

Using the technique of truncating the Painleve series expansion at different orders we obtained the exact travelling wave solution of the Boussinessq equation. Here we used and assumed solitary wave solution and searched for it where we obtained a special solution of the assumed form, since it exist. The method of the Painleve analysis is algorithmic and gives detail on the integrability aspect of the equation also[7].

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