The International Journal Of Engineering And Science (IJES)
|| Volume || 4 || Issue || 7 || Pages || PP.26-30|| 2015 ||
ISSN (e): 2319 – 1813 ISSN (p): 2319 – 1805

On Ternary Quadratic Equation

$$3(x^2+y^2) \quad - 5xy = 11\, z^2$$

M.A.Gopalan[1], R.Anbuselvi[2] , N.Ahila[3]

[1]*Department Of Mathematics, Shrimati Indira Gandhi College, Thiruchirappalli – 620 002, Tamil Nadu, India.*
[2]*Department Of Mathematics, A.D.M. College For Women (Autonomous), Nagapattinam – 600 001, Tamil Nadu, India.*
[3,] *Department Of Mathematics, Thiru.Vi. Ka. Govt. Arts College, Tiruvarur- 610003, Tamil Nadu, India.*

ABSTRACT

The Ternary Quadratic Diophantine Equation given by $3(x^2+y^2) - 5xy = 11\,z^2$ *is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.*

Keywords: *Ternary, Quadratic, Integral solutions.*

Date of Submission: 26 May 2015 Date of Accepted: 25 July 2015

I. INTRODUCTION

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety [1-5].For an extensive review of various problems, one may refer [6-20]. This communication concerns with yet another interesting ternary quadratic equation $3(x^2+y^2) - 5xy = 11\,z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

Notations Used

- $T_{m,n}$-Polygonal number of rank n with size m.
- P_n^k - Pentagonal number of rank n with size k.
- SqP_n - Square Pyramidal number of rank n.

Method Of Analysis

The Ternary Quadratic Diophantine Equation to be solved for its non-zero distinct integral solution is

$$3(x^2+y^2) - 5xy = 11z^2 \qquad (1)$$

The substitution of linear transformations$(u \neq v \neq 0)$

$$x = u+v,\; y = u-v \qquad (2)$$

in (1) leads to $u^2 = 11\left(z^2 - v^2\right)$ (3)

Different patterns of solutions of (1) are illustrated below

Pattern I

Equation (3) is equivalent to the system of double equations

$$uB - 11vA - 11zA = 0$$

$$uA + vB - zB = 0$$

This is satisfied by

$$u = 22AB;\; v = B^2 - 11A^2;\; z = B^2 + 11A^2$$

Hence in view of (2), the corresponding solutions of (1) are given by

$$x = x\,(A,B) = 22\,AB + B^2 - 11A^2$$
$$y = y\,(A,B) = 22\,AB - B^2 + 11A^2$$
$$z = z\,(A,B) = B^2 + 11A^2$$

A few interesting properties observed are as follows:

1. $x(2,B)+z(2,B)-128T_{3,B}+T_{126,B} \equiv 0 \pmod{83}$
2. $x\big(A,A(A+1)\big)+y\big(A,A(A+1)\big) = 88P_A^5$
3. $y(A,3)+z(A,3)-160T_{3,A}+T_{118,A} \equiv 0 \pmod{71}$
4. $y(B(B+1),B)+y(B(B+1),B)-44P_B^5+88T_{3,B}-T_{86,B} \equiv 0 \pmod{85}$
5. $y(A,5)-200T_{3,A}+178T_{3,A} \equiv -25 \pmod{89}$
6. Each of the following expressions represents a Nasty number
   a) $3\{x(b,-b)+3z(b,-b)\}$
   b) $6\{x(a,2a)-y(a,2a)+2z(a,2a)\}$
   c) $y(a,3a)-x(a,3a)+z(a,3a)$

It is observed that, by rewriting (3) suitably, one may arrive at the following patterns of solutions to (1).

Pattern II

Equation (3) is equivalent to the following equations

$$Bu-Av-Az=0$$

$$Au+11Bv-11Bz=0$$

From which we get

$$x = x\,(A,B) = 22\,AB+11B^2-A^2$$

$$y = y\,(A,B) = 22\,AB-11B^2+A^2$$

$$z = z\,(A,B) = 11B^2+A^2$$

A few interesting properties observed are as follows:

1. $x\big(A(A+1),(2A+1)\big)+y\big(A(A+1),(2A+1)\big) = SqP_A$
2. $y\big(A,A(A+1)\big)+z\big(A,A(A+1)\big)-44P_A^5-48T_{3,A}+T_{46,A} \equiv 0 \pmod{45}$
3. $x\big((B+1)(B+2),B\big)+z\big((B+1)(B+2),B\big)-132P_B^3-86T_{3,B}+T_{44,B} \equiv 0 \pmod{64}$
4. $x(2,B)-108T_{3,B}+T_{88,B} \equiv -4 \pmod{53}$
5. Each of the following expressions represents a Nasty number
   a) $3\{x(a,3a)+y(a,3a)-z(a,3a)\}$
   b) $x(2a,a)-3z(2a,a)$
   c) $3\{y(a,-a)+2z(a,-a)\}$

Pattern III

Equation (3) is equivalent to the following algebraic equations

$$Bu+11Av-11Av=0$$
$$Au-Bv-Bz=0$$

From which we get

$$x = x\,(A,B) = 22\,AB-B^2+11A^2$$

$$y = y\,(A,B) = 22\,AB+B^2-11A^2$$

$$z = z\,(A, B) = B^2 + 11A^2$$

A few interesting properties observed are as follows:

1. $x\big(A, A(A+1)\big) + z\big(A, A(A+1)\big) - 44P_A^5 - 96T_{3,A} + T_{54,A} \equiv 0 \pmod{73}$

2. $y\big((B+1)(B+2), B\big) + z\big((B+1)(B+2), B\big) - 132P_B^3 - 28T_{3,B} + T_{26,B} \equiv 0 \pmod{25}$

3. $x(A^2+1, A) - x(A^2-1, A) - 130T_{3,A} + T_{44,A} \equiv 0 \pmod{41}$

4. $x(A, 1) + y(A, 1) - z(A, 1) + 114T_{3,A} - T_{94,A} \equiv -1 \pmod{147}$

5. Each of the following expressions represents a Nasty number

a) $x(a, 3a) - y(a, 3a) + z(a, 3a)$
b) $y(b, b) + z(b, b)$
c) $3\{x(a, -a) + y(a, -a) + z(a, -a)\}$

Pattern IV

Equation (3) is equivalent to the following two equations

$$Bu + Av - Az = 0$$

$$Au - 11Bv - 11Bz = 0$$

From which we get

$$x = x\,(A, B) = 22\,AB - 11B^2 + A^2$$

$$y = y\,(A, B) = 22\,AB + 11B^2 - A^2$$

$$z = z\,(A, B) = 11B^2 + A^2$$

A few interesting properties observed are as follows:

1. $x(B(B+1), B) - z(B(B+1), B) - 44P_B^5 + 68T_{3,B} - T_{26,B} \equiv 0 \pmod{45}$

2. $x(A(A+1), A+2) + y(A(A+1), A+2) = 264P_A^3$

3. $x(3, B) - y(3, B) + z(3, B) + T_{88,B} - T_{66,B} \equiv 9 \pmod{11}$

4. $x(A, 1) - 20T_{3,A} + T_{20,A} \equiv -11 \pmod{4}$

5. Each of the following expressions represents a Nasty number

a) $4\{y(b, 2b) - 2z(b, 2b)\}$
b) $6\{x(a, a) - y(a, a) + 2z(a, a)\}$
c) $3(\{x(a, 3a) + y(a, 3a) - z(a, 3a)\})$

Pattern V

Assume $z = a^2 + 11b^2$ (4)

Where a, b are non-zero distinct integers.

Write 11 as $11 = \left(i\sqrt{11}\right)\left(-i\sqrt{11}\right)$ (5)

Use (4) and (5) in (3) and employing the method of factorization. Define

$$\left(u + i\sqrt{11}v\right) = \left(i\sqrt{11}\right)\left(a + i\sqrt{11}b\right)^2 \qquad (6)$$

Equating real and imaginary parts in (6) and using (2) ,the values of x and y satisfies (1) are given by

$$x = x(a, b) = -22ab + a^2 - 11b^2 \qquad (7)$$

$$y = y(a, b) = -22ab - a^2 + 11b^2 \qquad (8)$$

Thus (7), (8) and (4) represents non-zero distinct integral solutions of (1) in two parameters.

A few interesting properties observed are as follows:

1. $x(3, b) + z(3, b) - 168T_{3,b} + T_{166,b} \equiv 0 \pmod{231}$

2. $y(1, b) + z(1, b) - 108T_{3,b} + T_{66,b} \equiv 0 \pmod{107}$

3. $x\left(a, (a + 1)(a + 2)\right) + y\left(a, (a + 1)(a + 2)\right) = -264P_a^3$

4. $x(b(b + 1), b) - z(b(b + 1), b) + 44P_b^5 + 136T_{3,b} - T_{94,b} \equiv 0 \pmod{113}$

5. $y\left(a, (a + 1)(a + 2)\right) - y\left(a, (a + 1)(a + 2)\right) - 132P_a^3 - 46T_{3,a} - T_{44,a} \equiv 0 \pmod{43}$

6. Each of the following expressions represents a Nasty number

a) $x(a, -a) + z(a, -a)$

b) $6\{x(a, a) - y(a, a) + z(a, a)\}$

c) $x(b, 3b) - y(b, 3b) + z(b, 3b)$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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