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On uniformly continuous uniform space

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ABSTRACT

In this paper the sufficient conditions, for a uniform space to be a uniformly continuous space are determined. In particular it is proved that if there is a set K which is compact whose complement is uniformly isolated then the uniform space is uniformly continuous space. This also shows that if the set of all limit points of X is compact whose complement is uniformly isolated then the uniform space is uniformly continuous. It is also proved that the converse of the later statement is false by giving a counter example.

KEYWORDS: *Uniformly continuous space, uniformly isolated set.*

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Definition 1: Uniformly isolated set in metric space: In metric space X, a subset A of X is Uniformly isolated if there is $\in > 0$ *such that* $d(x,y) \geq \in \forall\, x \neq y \in A$.

Definition 2: Uniformly isolated set in Uniform space: Let (X,$\mathcal{U}$) be a uniform space. A subset A of X is Uniformly isolated if $(A \times A)^c \cup \Delta \in \mathcal{U}$.

We first show that the definition of uniformly isolated set coincides for a metric space which is also a uniform space.

Theorem1: If (X,d) is a metric space and $\mathcal{U}$ is a corresponding uniformity define by the metric d then A $\subset X$ is uniformly isolated in metric space (X,d) if and only if A is uniformly isolated set in uniform space (X,$\mathcal{U}$).

Proof: Let A $\subset X$ be any uniformly isolated set in metric space (X,d). *Then* $\exists \in > 0$ *such that* $\forall\, x \neq y \in A, d(x,y) \geq \in$. ie. For any $(x', y') \in A \times A - \Delta$, d$(x', y') \geq \in$ i.e. $A \times A - \Delta \subseteq \{(x,y)/\ d(x,y) \geq \in\}$. $\Longrightarrow$ $\{(x,y)/\ d(x,y) \geq \in\}^c \subseteq (A \times A - \Delta)^c = ((A \times A) \cap \Delta^c)^c = (A \times A)^c \cup \Delta$. ie. $\{(x,y)/\ d(x,y) \geq \in\}^c \subseteq (A \times A)^c \cup \Delta$. ie.{(x,y): d(x,y)< $\in$}= $V_{d,\epsilon} \subseteq (A \times A)^c \cup \Delta$. But $V_{d,\epsilon} \in \mathcal{U}$ *Hence* $(A \times A)^c \cup \Delta \in \mathcal{U}$. Thus if A is uniformly isolated set in a metric space then $(A \times A)^c \cup \Delta \in \mathcal{U}$ ie. A is uniformly isolated set in uniform space (X,$\mathcal{U}$).

Conversely: Let A be any uniformly isolated set in uniform space (X,$\mathcal{U}$). *Then*$(A \times A)^c \cup \Delta \in \mathcal{U}$. $\therefore$ $(A \times A)^c \cup \Delta$ is the union of members of the family $\{V_{d,r}: r > 0\}$. *ie.* $[(A \times A)^c \cup \Delta]^c \supset V_{d,r}$ for some r>0. ie. $A \times A - \Delta \subset {V_{d,r}}^c \Longrightarrow \forall\, x \neq y \in A, d(x,y) \geq r$. Thus if A is uniformly isolated set then $\exists \in > 0$ *such that* $d(x,y) \geq \in \forall\, x \neq y \in A$.

Theorem2: Let X be any Uniform space. If K is compact such that X-K is uniformly isolated then X is uniformly continuous space.

Proof: Iet f:X→ R be any continuous function. To show that f is uniformly continuous, Let $\in > 0$ be given. As X-K is uniformly isolated $[(X-K) \times (X-K)]^c \cup \Delta \in \mathcal{U}$ ie. V $\in \mathcal{U}$ where V = $[(X-K) \times (X-K)]^c \cup \Delta$. Since f is continuous on X, for every $x \in X \exists\, U_x \in \mathcal{U}$ such that y $\in U_x[x] \Longrightarrow |f(x) - f(y)| < \frac{\in}{2}$ ………..(1).
Since, "the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$". $\exists$ open symmetric member V_x of $\mathcal{U}$ such that $V_x \subset U_x$ and $V_x \circ V_x \subset U_x$. $\therefore$ $V_x[x']$ is open in X $\forall\, x' \in X$. Thus , $V_x[x]$ is open in X. ie for each x$\in X$ we get $V_x[x]$, open subset of X such that $V_x \circ V_x \subset U_x$. Since, $x \in V_x[x]\ \forall\, x \in X, K \subseteq \cup_{x \in K} V_x[x]$. Thus the family $\{V_x[x]: x \in K\}$ is an open cover for K and K is compact. $\therefore$ $\exists\, x_1, x_2 \ldots\ldots x_n$ in K such that K$\subseteq$ $\cup_{i=1}^n V_{x_i}[x_i]$. Put W= $\cap_{i=1}^n V_{x_i} \in \mathcal{U}$ Also $W_1 = W \cap V \in \mathcal{U}$. Now we show that (x,y) $\in W_1 \Longrightarrow |f(x) - f(y)| < \in$. Let (x,y) $\in W_1 \Longrightarrow$ (x,y) $\in W$ *and* (x,y) $\in V$=$[(X-K) \times (X-K)]^c \cup \Delta$. Now (x,y) $\in [(X-K) \times (X-K)]^c \Longrightarrow$ (x,y) $\notin (X-K) \times (X-K) \Longrightarrow x \notin X - K$ or $y \notin X - K \Longrightarrow x \in K$ or y $\in K$. If $x \in K \subseteq$

$\bigcup_{i=1}^{n} V_{x_i}[x_i]$, *then* x ∈ $V_{x_j}[x_j]$ for some j, $1 \leq j \leq n \Rightarrow (x, x_j) \in V_{x_j}$ for above j ………… (2). Also, (x,y) ∈ $W = \bigcap_{i=1}^{n} V_{x_i} \Rightarrow$ (x,y) ∈ V_{x_i} $\forall\, i = 1,2, \ldots. n.$ ∴ $(x, y) \in V_{x_j}$ for above j …………… (3). From (2) and (3) we get, $(x_j$, y)∈ $V_{x_j} \circ V_{x_j} \subset U_{x_j}$ ⟹ $(x_j$, y)∈ U_{x_j} for above j, $1 \leq j \leq n \Rightarrow \left|f(x_j) - f(y)\right| < \in/2$ for above j (by (1)) …………..(4).

From (2) $(x, x_j) \in V_{x_j}$ for above j ie $(x, x_j) \in V_{x_j} \subset V_{x_j} \circ V_{x_j} \subset U_{x_j} \Rightarrow (x, x_j) \in U_{x_j} \Rightarrow \left|f(x) - f(x_j)\right| < \in/2$ for above j ………….. (5).

∴ From (4) and (5) $|f(x) - f(y)|$ = $\left|f(x) - f(x_j) + f(x_j) - f(y)\right| \leq \left|f(x) - f(x_j)\right| + \left|f(x_j) - f(y)\right| < \frac{\in}{2} + \frac{\in}{2} = \in$. Similarly if y ∈ K then $|f(x) - f(y)| < \epsilon$. If (x,y)∈ Δ *then* $x = y$ ∴ $|f(x) - f(y)| < \in$ ie. For $\epsilon > 0\ \exists\, W_1 \in \mathcal{U}$ such that ∀ (x,y) ∈ $W_1 \Rightarrow |f(x) - f(y)| < \in$ ⟹ f is uniformly continuous on X. ⟹ X is uniformly continuous space.

Theorem3: Let X be a uniform space and A= set of all limit points of X. Suppose A is compact and X-A is uniformly isolated then X is uniformly continuous space.

Proof: Since A is compact, applying theorem (2) we get the result.

Remark: Converse of this theorem is false. We prove it by giving a counter example.

Ex. Let X=[-1,0]U{……..,$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$} U {2,3,4…….}. Then i]X is uniformly continuous space.

ii] A=[-1,0] is compact. iii]X-A is not uniformly isolated.

Sol: i] Take K=[-1,0]U{……..,$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$}. Then K is closed and bounded and hence compact. Using compact and for all $x, y \in X - K, x \neq y, d(x, y) \geq 1$ ∴X-K is uniformly isolated and by theorem1, X is uniformly continuous space.

ii] Since set of all limit points of X=A=[-1,0], A is compact.

iii] Now X-A= {……..,$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$} U {2,3,4…….}. In X-A we get a sequence {$\frac{1}{n}$} such that $\frac{1}{n} \to 0$ as $n \to \infty$. ∴ there does not exist $\delta > 0$ such that $x \neq y \Rightarrow d(x, y) > \delta$. ⟹ X-A is not unformly isolated.

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