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Information Matrices and Optimality Values for various Block Designs

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ABSTRACT
The conditions of an experiment allow the possibility of simultaneous existence of a number of experimental designs. To choose an appropriate design, is easy to analyze and satisfies optimal properties, developed by Kiefer (1959) based on the Information matrix of the experimental design. In this paper an attempt is made to obtain the information matrices for CRD, RBD, LSD and BIBD's and are illustrated in case of CRD, RBD and LSD and for different parameters of BIBD's.

Key words: *CRD, RBD, LSD, BIBD.*

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I. INTRODUCTION

An optimality criterion is a criterion which summarizes how good a design is and it is maximized or minimized by an optimal design. To estimate the treatment effects with a certain number of restrictions on allotment of treatments to the experimental units, and also to estimate how much information would have been lost by using a smaller number of restrictions, the optimality criteria can be used. Optimality criteria can be used to select best among the available designs belongs to a particular class of experimental design satisfying the conditions, and easy to analyze and are generally optimal only for a specific statistical model.

Kiefer (1959) developed a useful criterion for finding optimum designs based on the Information matrix. The information matrix is proportional to the inverse of the variance–covariance matrix of the least squares estimates of the linear parameters of the model. Some of the alphabetical optimality criteria are: A-, C-, D-, E-, G- optimality criteria.

DEFINITION 1.1: Let D be the class of designs, X be any design belongs to D , is said to be optimal if

$$\phi [\text{Var}(X)] \leq \phi [\text{Var}(X^*)] \qquad \text{for any } X, X^* \in D \qquad (1.1)$$

where ϕ is a criterion function of information matrix used for estimating the parameters in the model.

DEFINITION 1.2: A design $X \in D$ is said to be A-optimum in the class of designs D if

$$\text{Trace} [(X'X)^{-1}]_X \leq \text{Min} \{ \text{Trace} [(X'X)^{-1}]_{X^*} \} \qquad \text{for any } X, X^* \in D \qquad (1.2)$$

DEFINITION 1.3: A design $X \in D$ is said to be C-optimum in the class of designs D, if

$$[\lambda_1 / \lambda_n]_X \geq [\lambda_1 / \lambda_n]_{X^*} \geq 0 \qquad \text{for any } X, X^* \in D \qquad (1.3)$$

where λ_1 and λ_n are the largest and the smallest eigen values of X'X.

DEFINITION 1.4: A design $X \in D$ is said to be D-optimum in the class of designs D, if

$$| (X'X)^{-1} |_X \leq \text{Inf} | (X'X)^{-1} |_{X^*}. \qquad \text{for any } X, X^* \in D \qquad (1.4)$$

Where $| (X'X)^{-1} |$ indicates the Determinant of $(X'X)^{-1}$ matrix.

DEFINITION 1.5: A design $X \in D$ is said to be E-optimum in the class of designs D, if

$$\lambda_{\max [(X'X)^{-1}] X} \leq \lambda_{\max [(X'X)^{-1}] X^*}. \qquad \text{for any } X, X^* \in D \qquad (1.5)$$

DEFINITION 1.6: A design $X \in D$ is said to be G-optimum in the class of designs D if

$$\text{Min} \{ \text{Var}(\hat{Y}(x))_X \} \geq \text{Min Max} \{ \text{Var}(\hat{Y}(x))_{X^*} \} \quad \text{for any } X, X^* \in D \qquad (1.6)$$

2. INFORMATION MATRICES FOR VARIOUS DESIGNS

In this section an attempt is made to obtain information matrices and optimality values for the experimental designs, Completely Randomized Design, Randomized Block Design, Latin Square Design and Balanced Incomplete Block Design and are presented below with suitable examples.

2.1 COMPLETELY RANDOMIZED DESIGN**:** The statistical model of Randomized Block Design $y_{ij} = \mu + \alpha_I + \varepsilon_{ij}$ can be expressed in a general linear model as $\underline{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} + \underline{\boldsymbol{\varepsilon}}$, Where $\underline{\mathbf{Y}} = [Y_{11},..Y_{1j}, ..Y_{1n1} |\ldots| Y_{i1},..Y_{ij},\ldots Y_{in2} |\ldots| Y_{k1},..Y_{kj},\ldots Y_{knk}]'$ vector of responses, $\underline{\boldsymbol{\beta}} = [\mu \mid \alpha_1\ldots\alpha_i\ldots\alpha_k]'$ where μ is the mean, α_i is effect due to treatment (with 'k' treatments), $\underline{\varepsilon} = [\varepsilon_{11},.. \varepsilon_{1j}, .. \varepsilon_{1n1} |\ldots| \varepsilon_{i1}, ..\varepsilon_{ij},\ldots \varepsilon_{in2} | \ldots| \varepsilon_{k1},.. \varepsilon_{kj},\ldots \varepsilon_{knk}]'$ is the vector of random errors, follows $NI(0, \sigma^2)$ and X is the design matrix where

$$X_{N\times(1+k)} = \left[\begin{array}{c|ccccccc}
1 & 1 & 0 & . & 0 & 0 & 0 & . & 0 \\
1 & 1 & 0 & . & 0 & 0 & 0 & . & 0 \\
. & . & . & . & . & . & . & . & . \\
1 & 1 & 0 & . & 0 & 0 & 0 & . & 0 \\ \hline
1 & 0 & 1 & . & 0 & 0 & 0 & . & 0 \\
1 & 0 & 1 & . & 0 & 0 & 0 & . & 0 \\
. & . & . & . & . & . & . & . & . \\
1 & 0 & 1 & . & 0 & 0 & 0 & . & 0 \\ \hline
. & . & . & . & . & . & . & . & . \\ \hline
1 & 0 & 0 & . & 0 & 0 & 0 & . & 1 \\
1 & 0 & 0 & . & 0 & 0 & 0 & . & 1 \\
. & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & . & 0 & 0 & 0 & . & 1
\end{array}\right] \qquad X'X = \left[\begin{array}{c|cccc}
N & n_1 & n_2 & \ldots & n_k \\ \hline
n_1 & n_1 & 0 & 0 & 0 \\
n_2 & 0 & n_2 & 0 & 0 \\
. & . & . & . & . \\
n_k & 0 & 0 & 0 & n_k
\end{array}\right]_{(k+1)*(k+1)}$$

Note: Augment the conditions $\Sigma\ \alpha_I = 0$ in X under H_0 to estimate the parameters in the model.

2.2 RANDOMIZED BLOCK DESIGN: The general linear model for a Randomized Block Design is $\underline{\mathbf{Y}}=\mathbf{X}\underline{\boldsymbol{\beta}}+\underline{\boldsymbol{\varepsilon}}$ Where $\underline{\mathbf{Y}} = [Y_{11},..Y_{1j}, ..Y_{1n} |\ldots| Y_{i1},..Y_{ij},\ldots Y_{in} | \ldots| Y_{k1},..Y_{kj},\ldots Y_{kn}]'$ vector of responses, $\underline{\boldsymbol{\beta}} = [\mu \mid \alpha_1\ldots\alpha_i\ldots\alpha_k \mid \beta_1 \ldots \beta_j\ldots\beta_n]'$ where μ is the mean, α_i is effect due to i^{th} treatment, β_j is effect due to j^{th} block (with 'k' treatments and 'n' blocks), $\underline{\varepsilon} = [\varepsilon_{11},.. \varepsilon_{1j}, .. \varepsilon_{1n} |\ldots| \varepsilon_{i1}, ..\varepsilon_{ij},\ldots \varepsilon_{in} | \ldots| \varepsilon_{k1},.. \varepsilon_{kj},\ldots \varepsilon_{kn}]'$ is the vector of random errors and are follows $NI(0, \sigma^2)$ and X is the design matrix where

$$X = \begin{bmatrix}
1 & 1 & 0 & . & 0 & 1 & 0 & . & 0 \\
1 & 1 & 0 & . & 0 & 0 & 1 & . & 0 \\
. & . & . & . & . & . & . & . & . \\
1 & 1 & 0 & . & 0 & 0 & 0 & . & 1 \\
1 & 0 & 1 & . & 0 & 1 & 0 & . & 0 \\
1 & 0 & 1 & . & 0 & 0 & 1 & . & 0 \\
. & . & . & . & . & . & . & . & . \\
1 & 0 & 1 & . & 0 & 0 & 0 & . & 1 \\
. & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & . & 1 & 1 & 0 & . & 0 \\
1 & 0 & 0 & . & 1 & 0 & 1 & . & 0 \\
. & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & . & 1 & 0 & 0 & . & 1
\end{bmatrix} \qquad X'X = \begin{bmatrix}
nk_{1x1} & nJ_{1xk} & kJ_{1xn} \\
nJ_{kx1} & nI_{kxk} & J_{kxn} \\
kJ_{nx1} & J_{nxk} & kI_{nxn}
\end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix}
\dfrac{-1}{k+n+kn} & \dfrac{1+k}{k+n+kn}J_{1xk} & \dfrac{1+n}{k+n+kn}J_{1xn} \\
\dfrac{1+k}{k+n+kn}J_{kx1} & \left\{\dfrac{1}{n}-\dfrac{(1+k)^2}{k+n+kn}\right\}I_{kxk} & \left\{1-\dfrac{(1+k)(1+n)}{k+n+kn}\right\}J_{kxn} \\
\dfrac{1+n}{k+n+kn}J_{nx1} & \left[1-\dfrac{(1+k)^2}{k+n+kn}\right]J_{nxk} & \left\{\dfrac{1}{k}-\dfrac{(1+k)(k+n)}{k+n+kn}\right\}I_{nxn}
\end{bmatrix}_{(1+k+n)x(1+k+n)}$$

Note: Augment the conditions $\Sigma\alpha_i=0$, $\Sigma\beta_j=0$ in X under H_0 to estimate the parameters in the model.

2.3 LATIN SQUARE DESIGN: The general linear model for a Latin Square Design is $\underline{Y} = X\,\underline{\beta} + \underline{\varepsilon}$, where $\underline{Y} = [y_{111}\ldots y_{1jk}\ldots y_{1vv} | \ldots |y_{i11}\ldots y_{ijk}\ldots y_{ivv}| \ldots |y_{v11}\ldots y_{vjk}\ldots y_{vvv}]'$ is the vector of observations where y_{ijk} is the observation belongs to i^{th} row, j^{th} column and k^{th} treatment (design has 'v' rows, 'v' columns and 'v' treatments), $\underline{\beta} = [\mu \mid \alpha_1 \ldots \alpha_i \ldots \alpha_v \mid \beta_1\ldots \beta_j \ldots \beta_v \mid \gamma_1\ldots \gamma_k \ldots \gamma_v]'$ where μ is the mean α_i ,β_j , γ_k are the i^{th} row,

j^{th} column and γ_k is k^{th} treatment effects respectively. $\underline{\varepsilon} = [\varepsilon_{11\,1} \ldots \varepsilon_{1jk} \ldots \varepsilon_{1vv} | \ldots | \varepsilon_{i11} \ldots \varepsilon_{ijk} \ldots \varepsilon_{ivv} | \ldots | \varepsilon_{v11} \ldots \varepsilon_{vjk} \ldots \varepsilon_{vvv}]'$ is the vector of random errors follows NI(0, σ^2) and X is the design matrix where

$$X = \left[\begin{array}{c|cccc|cccc|cccc}
1 & 1 & 0 & . & 0 & 1 & 0 & . & 0 & 1 & 0 & . & 0 \\
1 & 1 & 0 & . & 0 & 0 & 1 & . & 0 & 0 & 1 & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 1 & 0 & . & 0 & 0 & 0 & . & 1 & 0 & 0 & . & 1 \\
\hline
1 & 0 & 1 & . & 0 & 1 & 0 & . & 0 & 0 & 1 & . & 0 \\
1 & 0 & 1 & . & 0 & 0 & 1 & . & 0 & 0 & 0 & . & 0. \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 0 & 1 & . & 0 & 0 & 0 & . & 1 & 1 & 0 & . & 0 \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
1 & 0 & 0 & . & 1 & 1 & 0 & . & 0 & 0 & 0 & . & 1 \\
1 & 0 & 0 & . & 1 & 0 & 1 & . & 0 & 1 & 0 & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & . & 1 & 0 & 0 & . & 1 & 0 & 0 & 1 & 0
\end{array}\right] \qquad X'X = \begin{bmatrix} N & vJ & vJ & vJ \\ vJ & vI & J & J \\ vJ & J & vI & J \\ vJ & J & J & vI \end{bmatrix}$$

Note: Augment the conditions $\Sigma\alpha_i=0$, $\Sigma\beta_j=0$, $\Sigma\gamma_k=0$ in X under H_0 to estimate the parameters in the model.

EXAMPLE 2.1: Consider an experimental design conducted in two way blocked design, the responses are presented below.

3.10 (C)	5.95 (F)	1.75 (A)	6.40 (E)	3.85 (B)	5.30 (D)
4.80 (B)	2.70 (A)	3.30 (C)	5.95 (F)	3.70 (D)	5.40 (E)
3.00 (A)	2.95 (B)	6.70 (E)	5.95 (D)	7.75 (F)	7.10 (C)
6.40 (E)	5.80 (D)	3.80 (B)	6.55 (C)	4.80 (A)	9.40 (F)
5.20 (F)	4.85 (C)	6.60 (D)	4.60 (B)	7.00 (E)	5.00 (A)
4.25 (D)	6.65 (E)	9.30 (F)	4.95 (A)	9.30 (C)	8.40 (F)

The optimality values are evaluated in case of CRD, RBD and LSD and are presented in Table 2.2 , by assuming the design as CRD by ignoring the blocking, RBD by ignoring the columns as blocks and LSD by consider the two way blocking.

	A- Opt	E- Opt	G- Opt	I- Opt
CRD	0.02040	7.8729	0.0029	0.0204
RBD	1.6875	-	0.0156	1.6875
LSD	2.5185	-	0.0123	2.518

Table 2.2

2.4 BALANCED INCOMPLETE BLOCK DESIGN: The General Linear Model for a Balanced Incomplete Block Design is **Y = Xβ + ε**. It can be expressed as :

$$
\begin{bmatrix} y_{11} \\ y_{12} \\ \ldots \\ y_{1b} \\ y_{21} \\ y_{22} \\ \ldots \\ y_{2b} \\ \ldots \\ y_{v1} \\ y_{v2} \\ \ldots \\ y_{vb} \end{bmatrix}_{vb\times 1}
=
\begin{bmatrix}
n_{11} & n_{11} & 0 & .. & 0 & n_{11} & 0 & \ldots & 0 \\
n_{12} & n_{12} & 0 & \ldots & 0 & 0 & n_{12} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n_{1b} & n_{1b} & 0 & \ldots & 0 & 0 & 0 & \ldots & n_{1b} \\
n_{21} & 0 & n_{21} & \ldots & 0 & n_{21} & 0 & \ldots & 0 \\
n_{22} & 0 & n_{22} & \ldots. & 0 & 0 & n_{22} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n_{2b} & 0 & n_{2b} & \ldots & \ldots & 0 & 0 & \ldots & n_{2b} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n_{v1} & 0 & 0 & \ldots & n_{v1} & n_{v1} & 0 & \ldots & 0 \\
n_{v2} & 0 & 0 & \ldots & n_{v2} & 0 & n_{v2} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n_{vb} & 0 & 0 & \ldots & n_{vb} & 0 & 0 & \ldots & n_{vb}
\end{bmatrix}_{vb\times(v+b+1)}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_v \\ \gamma_1 \\ \gamma_2 \\ \ldots \\ \gamma_b \end{bmatrix}_{(1+v+b)\times 1}
+
\begin{bmatrix} n_{11}\varepsilon_{11} \\ n_{12}\varepsilon_{12} \\ \ldots \\ n_{1b}\varepsilon_{1b} \\ n_{2b}\varepsilon_{2b} \\ n_{22}\varepsilon_{22} \\ \ldots \\ n_{2b}\varepsilon_{2b} \\ \ldots \\ n_{v1}\varepsilon_{v1} \\ n_{v2}\varepsilon_{v2} \\ \ldots \\ n_{vb}\varepsilon_{vb} \end{bmatrix}
$$

Then its X'X is
$$
X'X = \begin{bmatrix} bk & rJ_{1,v} & kJ_{1,b} \\ rJ'_{v,1} & rI_{v} & N \\ kJ'_{b,1} & N' & kI_{b} \end{bmatrix}
$$

The various optimality values for the different parameters of BIBD are evaluated and presented in the following Table 2.3.

No	**V**	**B**	**r**	**k**	**λ**	**E=λv/rk**	**A-opt**	**D-opt**	**E-opt**	**G-opt**	**I-opt**
1	4	6	3	2	1	0.666	1.6660	0.020800	6	0.416	1.66
2	4	4	3	3	2	0.888	3.1100	0.111100	9	0.77	3.11
3	9	12	4	3	1	0.750	2.7500	0.000012	12	0.3055	2.75
4	9	36	8	2	1	0.560	1.2050	0.000001	16	0.1338	1.205
5	9	12	3	4	1	0.750	4.0909	0.000355	11	0.4545	4.0909
6	9	12	8	6	5	0.930	2.6875	0.00000031	48	0.2986	2.6875
7	6	15	5	2	1	0.600	1.3500	0.000097	10	0.225	1.35
8	5	10	4	2	1	0.625	1.4580	0.001540	8	0.2916	1.458
9	6	15	10	4	6	0.900	1.2750	0.000024	40	0.2125	1.275
10	13	13	3	4	1	1.083	6.0660	0.000016	15	0.4666	6.066
11	13	13	4	4	1	0.812	4.06250	8503056	16	0.3125	4.0625
12	11	11	5	5	2	0.880	3.3730	0.000000	25	0.3066	3.373
13	5	10	4	2	1	0.625	1.4583	0.00154	8	0.2916	1.4583
14	5	5	4	4	3	0.937	4.0625	0.0625	16	0.8125	4.0625
15	5	10	6	3	3	0.833	1.3888	0.00068	18	0.2777	1.3888

16	6	15	6	2	1	0.500	1.350	0.00009	10	0.225	1.350
17	6	10	5	3	2	0.800	1.733	0.00027	15	0.2888	1.733
18	6	6	5	5	4	0.960	5.040	0.0400	25	0.840	5.040
19	6	6	5	5	6	1.440	4.971	0.0285	35	0.828	4.970
20	7	7	3	3	1	0.777	3.111	0.0017	9	0.4444	3.111
21	7	7	4	4	2	0.875	3.111	0.00097	16	0.4444	3.111
22	7	21	6	2	1	0.583	1.283	0.00001	12	0.1832	1.283
23	8	28	7	2	1	0.571	1.238	0.00007	14	0.154	1.238
24	8	14	7	4	3	0.857	1.785	0.0000021	28	0.2231	1.785
25	8	8	7	7	6	0.979	7.020	0.0200	49	0.8775	7.020
26	10	18	9	5	4	0.888	7.020	0.0000001	45	0.702	7.020
27	10	15	9	6	5	0.925	2.268	0.00000007	54	0.2268	2.2608
28	11	11	6	6	3	0.916	3.361	0.00000047	36	0.3055	3.361
29	11	55	1	2	1	0.550	1.161	0.000000000	20	0.1055	1.161
30	12	44	1	3	2	0.727	1.252	0.000000000	39	0.1043	1.252

Table 2.3

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