Integral Solutions of Binary Quadratic Diophantine

\[ x^2 + pxy + y^2 = N \]

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ABSTRACT

Non-trivial integral solutions for the binary quadratic diophantine equation \( x^2 + pxy + y^2 = N \), \( p > 2 \), \( N \not\equiv 0 \pmod{4} \) are obtained. The recurrence relations satisfied by the solutions along with a few examples are given.

KEY WORDS: Binary quadratic, integral solutions

MSC 2000 Subject classification number: 11D09

I. INTRODUCTION

It is well known that binomial quadratic (homogeneous or non-homogeneous) Diophantine equations are rich in variety[1,2]. The authors have considered the equation \( x^2 + xy + y^2 = N \), and analysed for its integer solutions[3]. In [4], non-trivial integral solutions for the binary quadratic diophantine equation \( x^2 + pxy + y^2 = 4N \) are obtained. In this communication, the non-trivial integral solutions for the binary quadratic diophantine equation \( x^2 + pxy + y^2 = N \), where \( p > 2 \) and \( N \not\equiv 0 \pmod{4} \) have been obtained. Also the recurrence relations among the solutions are given.

II. METHOD OF ANALYSIS

The equation to be solved is

\[ x^2 + pxy + y^2 = N \ , \ p > 2 \ , \ N \not\equiv 0 \pmod{4} \]  \( (1) \)

The substitution of the linear transformations

\[ u = X + (2 - p)T \]
\[ v = X - (2 + p)T \]

inequation (1) leads to

\[ 4X^2 = (p^2 - 4)4T^2 + N \]  \( (2) \)

Again, setting \( 2X = P, 2T = Q \)

Equation (2) becomes

\[ P^2 = (p^2 - 4)Q^2 + N \]  \( (3) \)

where \( p^2 - 4 \) is a square free non zero integer.
Assume that the initial solution of equation (2) be \((Q_0, P_0)\).

Consider the Pellian
\[
P^2 = (p^2 - 4)Q^2 + 1
\]
whose general solution \((\tilde{Q}_s, \tilde{P}_s)\) is given by
\[
\tilde{P}_s + \sqrt{p^2 - 4} \tilde{Q}_s = \left(\tilde{P}_0 + \sqrt{p^2 - 4} \tilde{Q}_0\right)^{s+1}, s = 0, 1, 2, \ldots
\]
in which \((\tilde{Q}_0, \tilde{P}_0)\) is the least positive integral solution of (4).

Applying Brahmagupta’s lemma, the sequence of solutions of equation (3) are given by
\[
\begin{align*}
Q_{s+1} &= P_0 \tilde{Q}_s + Q_0 \tilde{P}_s \\
P_{s+1} &= P_0 \tilde{P}_s + (p^2 - 4)Q_0 \tilde{Q}_s 
\end{align*}
\]
where \(s = 0, 1, 2, \ldots\)

The recurrence relations among the solutions are given by
\[
\begin{align*}
x_{s+1} &= \frac{1}{2} \left[ P_0 G + \sqrt{p^2 - 4 Q_0 F} \right] - \frac{p}{2 \sqrt{p^2 - 4}} \left[ P_0 F + \sqrt{p^2 - 4 Q_0 G} \right] \\
y_{s+1} &= \frac{1}{\sqrt{p^2 - 4}} \left[ P_0 F + \sqrt{p^2 - 4 Q_0 G} \right] 
\end{align*}
\]
where
\[
F = \left(\tilde{P}_0 + \sqrt{p^2 - 4} \tilde{Q}_0\right)^{s+1} + \left(\tilde{P}_0 - \sqrt{p^2 - 4} \tilde{Q}_0\right)^{s+1} \\
G = \left(\tilde{P}_0 + \sqrt{p^2 - 4} \tilde{Q}_0\right)^{s+1} + \left(\tilde{P}_0 - \sqrt{p^2 - 4} \tilde{Q}_0\right)^{s+1}
\]

Also the recurrence relations among the solutions are given by
\[
\begin{align*}
(1) \quad x_{s+3} - 2 \tilde{P}_0 x_{s+2} + x_{s+1} &= 0 \\
(2) \quad y_{s+3} - 2 \tilde{P}_0 y_{s+2} + y_{s+1} &= 0
\end{align*}
\]

To analyze the nature of solutions, one has to go in for particular values of \(p\) and \(N\). For the sake of simplicity and clear understanding, a few numerical examples are given below:
Illustration : 1.1

$p$ and $N$ are both odd.

Table 1.1(a)

$p = 3 \ N = 5$

\[
\begin{array}{|c|c|c|}
\hline
i & x_i & y_i \\
\hline
0 & -1 & 4 \\
1 & -29 & 76 \\
2 & -521 & 1364 \\
3 & -9349 & 24476 \\
4 & -167761 & 439204 \\
5 & -3010349 & 7881196 \\
6 & -54018521 & 141422324 \\
7 & -969323029 & 2537720636 \\
8 & -17393796001 & 45537549124 \\
9 & -312119004989 & 817138163596 \\
\hline
\end{array}
\]

Observations:

1. $y_i \equiv x_i \pmod{5}$

2. Each of the expressions $y_{2i} + x_{2i} - 2$ and $y_{2i+1} + x_{2i+1} + 2$ is a perfect square.

3. $x_i y_{i+1} - y_i x_{i+1} \equiv 0 \pmod{40}$

Illustration : 1.2

$p$ is even and $N$ is odd

Table 1.2(b)

$p = 4 \ N = 13$

\[
\begin{array}{|c|c|c|}
\hline
i & x_i & y_i \\
\hline
0 & 1 & 2 \\
1 & -9 & 34 \\
2 & -127 & 474 \\
3 & -1769 & 6602 \\
4 & -24639 & 91954 \\
5 & -343177 & 1280754 \\
6 & -4779839 & 17838602 \\
7 & -66574569 & 248459674 \\
8 & -927264127 & 3460596834 \\
9 & -12915123209 & 48199896002 \\
\hline
\end{array}
\]
Observations:
(1) \( y_{i+1} \equiv y_i \ (\text{mod} \ 8) \)
(2) \( y_{3i-1} - y_{3i-2} = 0 \ (\text{mod} \ 10) \)
(3) \( x_i + y_i \equiv x_i - y_i \ (\text{mod} \ 4) \)
(4) \( x_{i-1} \equiv x_i \ (\text{mod} \ 2) \)
(5) \( x_{i+1}y_i - y_{i+1}x_i + 52 = 0 \)

Illustration: 1.3
\( p \) and \( N \) are both even.

Table 1.3(c)

\[
p = 4 \quad N = 4
\]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
</tr>
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<td>0</td>
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Observations:
(1) \( x_{i+1}y_i - y_{i+1}x_i = 0 \ (\text{mod} \ 16) \)
(2) \( x_i + y_i \equiv x_i - y_i \ (\text{mod} \ 4) \)
(3) \( y_{i+1} \equiv y_i \ (\text{mod} \ 4) \)
(4) \( y_{3i-1} = y_{3i-2} \ (\text{mod} \ 4) \)
Illustration: 1.4

$p$ is odd and $N$ is even.

Table 1.4(d)

\[ p = 3 \quad N = 4 \]

<table>
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</tr>
</tbody>
</table>

Observations:

1. $x_{i+1}y_i - y_{i+1} = -104$
2. $y_j - x_i = y_i + x_j \mod 4$

In conclusion, one may search for other patterns of solutions and their corresponding properties.

REFERENCES: