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A note on estimation of parameters of multinormal distribution with constraints

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ABSTRACT

Mardia et al (1989) considered the problem of estimating the parameters of nonsingular multivariate normal distribution with certain constraints. Nagmur (2003) considered the problem of estimating the mean sub-vector of non-sigular multivariate normal distribution with certain constraints. In this paper we try to estimate mean sub-vector under some different constraints and submatrix of ∑ with certain constraints for a nonsingular multivariate normal distribution.

Keywords: *Likelihood Function, Maximum Likelihood Estimator, Non-singular Multivariate Normal Distribution, Constraints.*

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I. INTRODUCTION

Mardia et al (1989) considered the estimation of parameters of non-singular multivariate normal distribution with and without constraints. Nagnur (2003) tried to obtain the Mle of sub mean vector of the distribution which can be useful in some practical problems. Mardia et al (1989) considered two type of constraints on the mean vector $\underline{\mu}$.

i. $\underline{\mu} = k\underline{\mu}_0$ i.e. $\underline{\mu}$ is known to be proportional to a known vector $\underline{\mu}_0$. For example, the elements of $\underline{x}$ could represent a sample of repeated measurements, in which case $\underline{\mu} = k\underline{1}$

ii. Another type of constraint is $R\underline{\mu} = \underline{r}$, where R and r are pre-specified

The first type of constraint was considered by Nagnur (2003) for sub mean vector of the distribution. In this paper, we consider type of constraint for sub mean vector and give the explicit expression for the estimators.

Mardia et al (1989) also considered constraint on variance-covariance matrix Σ, viz Σ=kΣ0 where Σ0 is known. We consider constraint on sub matrix of Σ and obtain its estimator with constraints.

2. ML Estimator of μ :

Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ be n iid observation from $N_p\left(\underline{\mu},\Sigma\right)$ population, where Σ is positive definite matrix. Suppose that $\underline{x}$, « and Σ are partitioned as follows :

$$\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix}, \qquad \underline{\mu} = \begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Σ_{11}: r x r, Σ_{22} : s x s with r + s = p. Let $\overline{x}$ and $S = \sum_{i=1}^{n}\left(\underline{x}_i - \overline{\underline{x}}\right)\left(\underline{x}_i - \overline{\underline{x}}\right)'$ be the mean vector and SSP matrix based on the sample observations $x_1, x_2, \ldots\ldots\ldots x_n$. The sample mean vector and SSP matrix is partitioned as

$$\underline{\bar{x}} = \begin{pmatrix} \underline{\bar{x}}_1 \\ \underline{\bar{x}}_2 \end{pmatrix} \qquad\qquad S = \begin{pmatrix} S_{11} & \mathrm{S}_{12} \\ \mathrm{S}_{21} & S_{22} \end{pmatrix}$$

where S_{11} : r x r , S_{22} : s x s ,with r + s = p.

Our problem is to estimate $\underline{\mu}$: r $\times$ 1 under the constraint R$\underline{\mu}_1$ = $\underline{r}$, where R and $\underline{r}$ are pre-specified. Maximizing the log likelihood subject to this constraint may be achieved by augmenting the log likelihood with a Lagrangian expression, i.e. we maximize

$$L^+ = L - n\,\underline{\lambda}'\,(R\underline{\mu} - \underline{r}) \tag{2.1}$$

where λ is a vector of Lagrangian multipliers and L is given by

$$L = -\frac{np}{2}\log 2\pi - \frac{n}{2}\log|\Sigma| - \frac{1}{2}tr\Sigma^{-1}\,S - \frac{n}{2}\left(\underline{\bar{x}} - \underline{\mu}\right)'\,\Sigma^{-1}\,\left(\underline{\bar{x}} - \underline{\mu}\right) \tag{2.2}$$

Case-1 (ρ Known)

With Σ assumed to be known, to find m.l.e.'s of $\underline{«}_1$ we are required to find λ for which the solution to $\partial L^+ / \partial \underline{«}_1 = \underline{0}$ satisfies the constraint R $\underline{«}_1$ = $\underline{r}$**.**

Observe that L^+ can be expressed as

$$L^+ = -\frac{np}{2}\log 2® - \frac{n}{2}\log|\Sigma| - \frac{1}{2}tr\Sigma^{-1}S - \frac{n}{2}\left\{\left(\underline{\bar{x}} - \underline{«}_1\right)'\,\Sigma^{11}\left(\underline{\bar{x}} - \underline{«}_1\right)\right\}$$
$$+\frac{n}{2}\left\{\left(2\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right)'\Sigma^{12}\left(\underline{\bar{x}}_2 - \underline{\mu}_2\right) + \left(\underline{\bar{x}}_2 - \underline{\mu}_2\right)\Sigma^{22}\left(\underline{\bar{x}}_2 - \underline{\mu}_2\right)' - n\lambda'\left(R\underline{\mu}_1 - \underline{r}\right)\right)\right\} \tag{2.3}$$

Now

$$\frac{\partial L^+}{\partial \underline{\mu}_1} = n\Sigma^{11}\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right) + n\Sigma^{12}\left(\underline{\bar{x}}_2 - \underline{\mu}_2\right) - nR'\underline{\square} = 0 \tag{2.4}$$

and

$$\frac{\partial L^+}{\partial \underline{\mu}_2} = n\Sigma^{22}\left(\underline{\bar{x}}_2 - \underline{\mu}_2\right) + n\Sigma^{21}\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right) = 0 \tag{2.5}$$

From (2.5) we get

$$\left(\underline{\bar{x}}_2 - \underline{«}_2\right) = -\left(\Sigma_{22}\right)^{-1}\Sigma^{21}\left(\underline{\bar{x}}_1 - \underline{«}_1\right) \tag{2.6}$$

Substituting for $\left(\underline{\bar{x}}_2 - \underline{«}_2\right)$ in (2.4), we get

$$n\Sigma^{11}\left(\underline{\bar{x}}_1 - \underline{«}_1\right) - n\Sigma^{12}\left(\Sigma^{22}\right)^{-1}\Sigma^{21}\left(\underline{\bar{x}}_1 - \underline{«}_1\right) - nR'\text{a} = 0$$

Since

$$\Sigma^{11} - \Sigma^{12}\left(\Sigma^{22}\right)^{-1}\Sigma^{21} = .\Sigma_{11}^{-1}.\,.,$$

we get $n\Sigma_{11}^{-1}\left(\underline{\bar{x}}_1 - \underline{«}_1\right) = nR'\underline{\text{a}}..$ **(2.7)**

Thus $\left(\underline{x}_1 - \underline{«}_1\right) = \sum_{11} R'\text{a}$,

Pre-multiplying by R gives $\left(R\underline{\bar{x}}_1 - \underline{r}\right) = \left(R\Sigma_{11}R'\right)\underline{\text{a}}$ if the constraint $R\underline{«}_1 = \underline{r}$ is to be satisfied. Thus, we take,

$$\underline{\text{a}} = \left(R\Sigma_{11}R'\right)^{-1}\left(R\underline{\bar{x}}_1 - \underline{r}\right) \text{ ,so}$$

$$\underline{\hat{«}}_1 = \underline{\bar{x}}_1 - \Sigma_{11}R'\underline{\text{a}} \qquad \textbf{(2.8)}$$

From (6), the ML estimator of $\underline{«}_2$ is

$$\underline{\hat{«}}_2 = \underline{\bar{x}}_2 + \left(\Sigma^{22}\right)^{-1}\Sigma^{21}\left(\underline{\bar{x}}_1 - \underline{\hat{«}}_1\right) \qquad \textbf{(2.9)}$$

Case 2 ($\sum$ unknown) :

When the covariance matrix $\sum$ is not known, we have to estimate $\underline{«}_1, \underline{«}_2$ and $\sum$ using the likelihood function (2.3) The ML estimator of $\sum$ is

$$\hat{\Sigma} = \frac{S^*}{n} \text{ , where } S^* = \sum_{i-1}^{n}\left(\underline{x}_i - \underline{\hat{\mu}}\right)\left(\underline{x}_i - \underline{\hat{\mu}}\right)' \qquad \textbf{(2.10)}$$

and $\underline{\hat{\mu}}$ is the ML estimator of $\underline{«}$ under the restriction R $\underline{\mu}_1$ = **r.**

To estimate ($\underline{«}_1, \underline{«}_2$) the likelihood equations are given by (2.4) and (2.5). Now we have

$$\hat{\Sigma} = \frac{S^*}{n} = \frac{S}{n} + \left(\underline{\bar{x}} - \underline{\hat{\hat{\mu}}}\right)\left(\underline{\bar{x}} - \underline{\hat{\hat{\mu}}}\right)' \qquad \textbf{(2.11)}$$

From (2.11), we have

$$I = \hat{\Sigma}^{-1} S/n + \hat{\Sigma}^{-1}\left(\underline{\bar{x}} - \underline{\hat{\hat{\mu}}}\right)\left(\underline{\bar{x}} - \underline{\hat{\hat{\mu}}}\right)' \qquad \textbf{(2.12)}$$

where $\underline{\hat{\hat{«}}} = \left(\underline{\hat{\hat{«}}}_1, \underline{\hat{\hat{«}}}_2\right)$ is the solution of (2.4) and (2.5) after replacing p^{ij} by $\hat{\sum}^{ij}$.

Since $\underline{«}_1$ and $\underline{«}_2$ have to satisfy these equations, from (2.12) we get the equations

$$\underline{\mu}' = \underline{\mu}_1{}'\hat{\Sigma}^{11} S_{11}/n + \mu_1{}'\hat{\Sigma}^{12} \; S_{21}/n \qquad \textbf{(2.13)}$$

$$O = \hat{\Sigma}^{21} S_{11}/n + \hat{\Sigma}^{22} S_{21}/n \qquad \textbf{(2.14)}$$

From above equations it follows that

$$\underline{«}_1{}'\left(S_{11}\big/n\right)^{-1} = \underline{«}_1{}'\left(\hat{\Sigma}^{11} - \hat{\Sigma}^{12}\left(\Sigma^{22}\right)^{-1}\hat{\Sigma}^{21}\right)$$

$$= \underline{«}_1{}'\,\hat{\Sigma}_{11}^{-1} \qquad \textbf{(2.15)}$$

Hence, from (2.7), we have

$$\left(\underline{\bar{x}}_1 - \underline{«}_1\right) = S_{11}R'\,\underline{ª}/n \qquad \textbf{(2.16)}$$

Pre multiplying (2.16) by R we get

$$\left(R\underline{\bar{x}}_1 - \underline{r}\right) = \left(RS_{11}R'\right)\underline{ª}/n \qquad \textbf{(2.17)}$$

provided the constraints $R\underline{\mu}_1 = \underline{r}$ is to be satisfied.

Thus we have

$$\underline{ª} = n\left(RS_{11}R'\right)^{-1}\left(R\underline{\bar{x}}_1 - \underline{r}\right) \qquad \textbf{(2.18)}$$

$$\hat{\hat{\underline{\mu}}}_1 = \underline{\bar{x}}_1 - S_{11}R'\left(RS_{11}R'\right)^{-1}\left(R\underline{\bar{x}}_1 - \underline{r}\right) \qquad \textbf{(2.19)}$$

The ML estimator of $\underline{\mu}_2$ is

$$\hat{\hat{\underline{\mu}}}_2 = \underline{\bar{x}}_2 + \left(S^{22}\right)^{-1} S^{21}\left(\underline{\bar{x}}_1 - \hat{\hat{\underline{\mu}}}_1\right)$$

3. ML Estimators of ∑ :

According to Mardia et al (1989), the likelihood function of p-variate normal distribution with constraints $\Sigma = k\Sigma_0$, where Σ_0 is known, is

$$2n^{-1}l\left(x, «, k\right) = -p\log k - \log\left|2®\Sigma_0\right| - k^{-1}\sim \qquad \textbf{(3.1)}$$

where $\alpha = tr\Sigma_0^{-1}S + \left(\underline{\bar{x}} - \underline{\mu}_0\right)'\Sigma_0^{-1}\left(\underline{\bar{x}} - \underline{\mu}_0\right)$ is independent of k.

Our problem is to obtain estimate of k for the constraint $p_{11} = kp_{110}$, where p_{110} is known and

p_{11}: r x r is a submatrix of p. If we let $p_0 = \begin{pmatrix} kp_{110} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, then

$$|\Sigma_0| = k^r |\Sigma_{110}||\Sigma_{22}||\Sigma_{21}\Sigma_{110}^{-1}\Sigma_{12}\Sigma_{22}^{-1}|\left(\sum_{j=0}^{s} C_j \left(\frac{-1}{k}\right)^{s-j}\right) \quad \textbf{(3.2)}$$

where $C_j = tr_j \left(p_{21}p_{110}^{-1}p_{12}p_{22}^{-1}\right)^{-1}$

Further, by considering p_0^{-1} as

$$\Sigma_0^{-1} = \begin{pmatrix} \Sigma_{110.2}^{-1} & \frac{-1}{k}\Sigma_{110}^{-1}\Sigma_{12}\Sigma_{22.10}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{110.2}^{-1} & \Sigma_{22.10}^{-1} \end{pmatrix}$$

where

$$p_{110.2} = kp_{110} - p_{12}p_{22}^{-1}p_{21}, \quad \Sigma_{22.10} = \Sigma_{22} - \frac{1}{k}\Sigma_{21}\Sigma_{110}^{-1}\Sigma_{12}; \quad \text{we get}$$

$$tr\Sigma_0^{-1}S = tr\Sigma_{22}^{-1}S_{22} + \frac{1}{k}tr\left(\Sigma_{110}^{-1}S_{11} - \Sigma_{110}^{-1}\Sigma_{12}\Sigma_{22}^{-1}S_{21} - \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{110}^{-1}S_{12} - \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{110}^{-1}S_{22}\right) + O\left(k^{-2}\right)$$

(3.3)

Since, $\Sigma_{110.2}^{-1} = \frac{1}{k}\Sigma_{110}^{-1} + O\left(k^{-2}\right)$ **(3.4)**

and

$$\Sigma_{22.10}^{-1} = \Sigma_{22}^{-1} - \frac{1}{k}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{110}^{-1} + O\left(k^{-2}\right), \quad \textbf{(3.5)}$$

We have

$$\left(\underline{\bar{x}} - \mu\right)'\Sigma_0^{-1}\left(\underline{\bar{x}} - \mu\right) =$$

$$\left(\underline{\bar{x}}_2 - \underline{\mu}_2\right)\Sigma_{22}^{-1}\left(\underline{\bar{x}}_2 - \underline{\mu}_2\right)' - \tfrac{1}{k}\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right)'\Sigma_{11}^{-1}\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right) + \left(\underline{\bar{x}}_2 - \underline{\mu}_2\right)'\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{110}^{-1}\left\{2\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right) + \left(\underline{\bar{x}}_2 - \underline{\mu}_2\right)\right\} + O\left(k^{-2}\right)$$

(3.6)

Using results (3.2), (3.3) and (3.6) in (3.1) we get

$$l^+ = 2n^{-1}l\left(x, \underline{«}, k\right) = Const. - \left(p + r\right)\log k - \frac{1}{k}\sim * + O\left(k^{-2}\right) \quad \textbf{(3.7)}$$

where

$$Const = -p\log 2® - \log|p_{110}| - \log|p_{22}| - 2\log|p_{21}p_{110}^{-1}p_{12}p_{22}^{-1}|,$$

$$\sim * = b_{s-1} + tr\Sigma_{22}^{-1}S_{22} + \left(\underline{\bar{x}}_2 - \underline{«}_2\right)'\Sigma_{22}^{-1}\left(\underline{\bar{x}}_2 - \underline{«}_2\right)$$

and

$$b_{s-1} = \frac{c_{s-1}}{c_s} = \frac{tr_{s-1}\left(\Sigma_{21}\Sigma_{110}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\right)^{-1}}{\left|\left(\Sigma_{21}\Sigma_{110}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\right)^{-1}\right|} \quad \textbf{(3.8)}$$

Thus, if $\underline{\mu}$ is known, then the mle of k is

$$\hat{k} = \frac{\sim^*}{p + r} \quad \textbf{(3.9)}$$

If $\underline{\mu}$ is unknown and unconstrained, then the mle's of $\underline{\mu}$ is $\underline{\bar{x}}$ and that of k is $\sim^{*}/p+r$, together these gives

$$\hat{k} = \frac{b_{s-1} + tr\Sigma_{22}^{-1}S_{22}}{p+r} \tag{3.10}$$

Note that for the constraint $\mathsf{p}_{22} = k\mathsf{p}_{220}$, where p_{220} is known, the mle's of $\hat{k}$ for known $\underline{\mu}$ is $/(p+s)$ and that of for unknown $\underline{\mu}$ is $^{*}/(p+s)$ where

$$\beta = d_{r-1} + tr\Sigma_{11}^{-1}S_{11} + \left(\underline{\bar{x}}_1 - \underline{\mu}_1\right)'\Sigma_{11}^{-1}\left(\underline{\bar{x}}_1 - \underline{\mu}_1\right), \quad {}^{*} = d_{r-1} + tr\Sigma_{11}^{-1}S_{11} \text{ with}$$

$$d_{r-1} = \left|\mathsf{p}_{12}\mathsf{p}_{22}^{-1}\mathsf{p}_{21}\mathsf{p}_{11}^{-1}\right| tr_{r-1}\left(\mathsf{p}_{12}\mathsf{p}_{22}^{-1}\mathsf{p}_{21}\mathsf{p}_{11}^{-1}\right)$$

4. Illustrative Example :

1. Estimation of sub mean vector with constraint $R\underline{\mu}_1 = \underline{r}$.

For 47 female cats the body weights (kgs), heart weights (gms), lungs weights (gms) and Kidney weights (gms) were recorded. The sample mean vector and covariance matrix are

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{pmatrix} = \begin{pmatrix} \underline{\bar{x}}_1 \\ \underline{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} 2.36 \\ 9.20 \\ 3.56 \\ 2.76 \end{pmatrix}$$

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} 0.0735 & 0.1937 & 0.2156 & 0.3072 \\ 0.1937 & 1.8040 & 0.1969 & 0.1057 \\ 0.2156 & 0.1969 & 2.1420 & 0.1525 \\ 0.3072 & 0.1057 & 0.1525 & 2.0136 \end{pmatrix}$$

Note that here $\underline{\bar{x}}$ and S are unconstrained mle's of $\underline{\mu}$ and Σ. However, from other information we know that

$$« = \begin{pmatrix} «_1 \\ «_2 \\ «_3 \\ «_4 \end{pmatrix} = \begin{pmatrix} \underline{«}_1 \\ \underline{«}_2 \end{pmatrix} = \begin{pmatrix} 2\,.5 \\ 10\ \ .0 \\ 3\,.5 \\ 2\,.5 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} 0.07810 & 0.15620 & 0.21745 & 0.31033 \\ 0.15620 & 1.56200 & 0.17767 & 0.07322 \\ 0.21745 & 0.17767 & 2.15436 & 0.17338 \\ 0.31033 & 0.07322 & 0.17338 & 2.04885 \end{pmatrix}$$

With above given information we estimate sub-mean vector $\underline{«}_1 = 2\times 1$ under the constraint $R\underline{«}_1 = \underline{r}$ where R : 2 x 2 and $\underline{r}$: 2 x 1 are pre-specified as follows :

$$R = \begin{pmatrix} 0.45000 & 0.12500 \\ 0.12500 & 0.91875 \end{pmatrix} \textit{and} \qquad \underline{r} = \begin{pmatrix} 2.37500 \\ 9.50000 \end{pmatrix}$$

For unknown Σ, we have

$$\underline{\text{a}} = n\left(R\text{S}_{11}\,\text{R}'\right)^{-1}\left(R\underline{\bar{x}}_1 - \bar{r}\right) = \begin{pmatrix} -2.25821 \\ -0.056431 \end{pmatrix}$$

$$\hat{\hat{\text{«}}}_1 = \underline{\bar{x}}_1 - S_{11}R'\underline{\text{a}} = \begin{pmatrix} 2.49993 \\ 10.00096 \end{pmatrix}$$

and $$\underline{\hat{\hat{\text{«}}}}_2 = \underline{\bar{x}}_2 + \left(S^{22}\right)^{-1} S^{21}\left(\underline{\bar{x}}_1 - \hat{\hat{\text{«}}}_1\right)$$

$$= \underline{\bar{x}}_2 - S_{21}S_{11}^{-1}\left(\underline{\bar{x}}_1 - \hat{\hat{\text{«}}}_1\right)$$

$$= \begin{pmatrix} 3.84642 \\ 3.10968 \end{pmatrix}$$

For known Σ, we have

$$\underline{\text{a}} = \begin{pmatrix} -2.0804 \\ -0.1706 \end{pmatrix}, \underline{\hat{\text{«}}}_1 = \begin{pmatrix} 2.4999 \\ 10.0006 \end{pmatrix}, \underline{\hat{\text{«}}}_2 = \begin{pmatrix} 3.9219 \\ 3.1205 \end{pmatrix}$$

2. Estimation of sub-matrix Σ_{11} of Σ with constraint $\Sigma_{11} = k\,\Sigma_{110}$.

For $\Sigma_{110} = \begin{pmatrix} 0.1 & 0.2 \\ 0.2 & 2.0 \end{pmatrix}$, we have for known $\underline{\mu}$ $\hat{k}$ = 0.49426 and for unknown $\underline{\mu}$, we have $\hat{\hat{k}}$ = 0.4886

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