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Duality Theory in Multi Objective Linear Programming Problems

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ABSTRACT

This work is aimed at applying duality theory to multi-objective linear programming problems. Three methods of solutions were discussed and applied in this study with the sole aim of finding out which of these three methods is the best for solving Multi Objective Linear Programming (MOLP) problems. The major components of the models used are the weighted sum and the e-constraint terms. Dual optimal functions were used in performing sensitivity analyses. The investigation in this study was carried out using a bank's investment data whose dual results showed that the merged weighted sum/e-constraint approach is the best method for handling MOLP problems

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I. INTRODUCTION

The theory of duality is one of the most important and interesting concepts in both single objective and multiple objective mathematical programming of operations research. Our focus in this work is to apply duality theory to multiple objective linear programming. The basic idea behind the duality theory is that every primal linear programming problem has an associated linear programming problem called its dual such that a solution to the primal linear programming also provides a solution to its dual. In view of the above discussion, the problem we are looking at in this study is a multiple objective-programming problem and we will be adapting appropriate duality theorems with which to solve it. In a multi objective linear programming problem (MOLP), each feasible solution set is said to be an efficient solution since such a solution can not be optimal to all the objective functions at the same time.

Philip (1972) stated that there is always a dual solution for each efficient primal solution and that the dual variables are the "prices" of the resources of the problem which are dependent on the subjective weights of the primal problem. Kornbluth (1974) investigated the duality of multiple objective linear programmes and used the dual optimal function to perform sensitivity analysis. Lohne (1990) developed a duality theory for linear multiple objective programming problem in which he was able to verify similar properties as in the scalar case, that is, single objective linear programming by using what is called "strong proper optima". In the study he found out that such optima and its associated dual solution are characterized by means of complementary slackness conditions. Arua *et al* (2000) observed that primal – dual programming problems can either be symmetric or asymmetric. Heyde *et al* (2006) developed a geometric approach to duality in multiple objective linear programming which is based on the duality of polytopes with one – to – one mapping between the minimal faces of the image of the primal objective function and the maximal faces of the image of the dual objective function.

Klamroth *et al* (2003) studied a linear multiple criteria optimization and specified that weighted sum approach and e-constraint approach can be used to solve the problem. They asserted that duality is a powerful tool for the generation of weighting vectors and hence of utility functions for MOP. They went further to indicate that the weighting vectors can be deduced from the optimal dual variables of the e-constraint problems. They asserted that weighted sum secularization of the objective functions cannot be used to generate all non dominated solutions because some will be supported while others will be unsupported. The above discussions from the earlier works in the literature makes it clear that duality theory can be applied to multiple objective linear programming when the problem is reduced from MOP to a single objective problemm. In view of this, we modify approach adopted in by Klamroth et al (2003) by combining weighted sum approach and e -constraint in solving a multiple objective programming problem. Furthermore the same problem was solved using weighted and e-constraint method separately so as to find out which approach is better.

II. METHODOLOGY

Duality generally follows from the main (primal) linear programming problem (mathematical program whose objective function(s) and constraints are in linear form.), which can either, be a maximization or minimization problem as the case may be. In many applications of linear and mathematical programming, it is often difficult or somehow misleading to represent the aspirations of the decision maker (or decision making body) in terms of a single objective function because of their diverse nature of activities. Indeed in complex industrial or governmental problems, the decision maker may frequently have to choose between alternative courses of action on the basis of multiple and conflicting criteria. Hence, MOLP is of mathematical programming that deals with decision problems characterized by multiple and conflicting objective functions that are to be optimized over a feasible set of decisions. Such problems referred to as multi-objective programs are commonly encountered in many areas of human activities including engineering, management and others. This concept began with the work of Charnes and Cooper in 1961 and has been refined and extended by many other researchers since then.

MOLP PROBLEM FORMULATION

We use the following notation throughout this work. x denotes x_i for all i = 1, …, k.
For k x n matix, $C_{i.} = C_i$; i.e ith row of C.
N_j is submatrix consisting of the rows in J_i – constraints which form the e-constraint equations.
Hence $f_j(x) = N_j(x)$
We consider the following general multiple objective program
(MOP).

$$\left.\begin{aligned} &\text{Max } \{ Z_1 = f_1(x) \} \\ &\quad \vdots \\ &\text{Max } \{ Z_k = f_k(x) \} \end{aligned}\right\} \qquad (1)$$

Subject to x ϵ S (Klamroth, 2003)
Where $S \subseteq R^n$ is the feasible set consisting of $x_1, x_2, \ldots x_n$ and $f_i(x)$, i = 1, 2, ….., k, are real-valued functions.
The problem above is called a multiple objective linear program (MOLP) if

$f_i(x) = C_i x \quad \forall\ i = 1, 2, \ldots, k$ and
$S = \{ x \in R^n;\ Ax \le b;\ x \ge 0 \}$

Where we assume that the vectors C= (C_i) ί =1,2…., k , b , b m-components and the m x n – matrix A are all real valued. Thus, the MOLP can be written as
Max Z = Cx
Subject to $Ax \le b;\ x \ge 0$

We define the set of all feasible criterion vectors Z, the set of all non-dominated criterion vectors N and the set of all efficient points E of equation (1) as follows:-

$$\left.\begin{aligned} &Z = \{ z \in R^k : z = f(x), x \in S\} = f(S) \\ &N = \{z \in Z : \not\ni \bar{z} \in Z : \bar{z} \ge z\} \\ &E = \{x \in S : f(x) \in N\} \\ &\text{Where } f(x) = (f_1(x), \ldots, fk(x))^T \end{aligned}\right\} \qquad (2)$$

GENERATION OF THE SOLUTION SETS

There are two general approaches to generate solution sets of MOPs. They are scalarization methods and non-scalarization methods. Scalarization method involves formulating an MOP – related SOP which is a function of the objective functions of the MOP while non scalarization methods do not explicitly use a scalarization function. To solve the MOP problem in this work, we used scalarization methods. The two well known scalarization techniques used in this work include the weighted sum approach and e-constraint approach. The third approach is gotten by merging these two approaches. These three approaches will be used to solve the same problem so as to know the one that is the best for handling MOLP problems.

WEIGHTED-SUM APPROACH : In the weighted sum approach, a weighted sum of the objective functions is maximized or minimized as the case may be. These objective functions of MOLP are pooled together using the strictly positive weighting vectors say $\{\hat{u} \in R^k: \hat{u} > 0, \Sigma \hat{u}_i = 1\}$, such that the MOLP becomes SOP given by

$$\text{Max } z = \hat{u}^T C(x)$$
$$\text{Subject to } Ax \leq b;\ x \geq 0$$

e-CONSTRAINT APPROACH : In the e-constraint approach, one objective function out of the objectives of the MOLP is retained as a scalar – valued objective while all the other objective functions are made new constraints by assigning goals/targets (to be achieved) to them. Hence, the i^{th} objective e-constraint program becomes:

$$\text{Max } z = C_i(x);$$
$$\text{Subject to } N_j(x) \geq e_j\ \forall_j \in Ji\ ;$$
$$Ax \leq b;\ x \geq 0$$

Where: $C_i x$ is one out of the given objective functions

$N_j x \geq e_j$ = the remaining objective functions (with the targets, e_j assigned to them) which now form part of the constraint equations. Hence they are called e-constraint equations.

The Merged Weighted Sum/ e-Constraint Approach : In MOLP, goals could be set by the decision maker and such goals can either be overachieved or underachieved. Usually interest is to achieve goals set in any firm and in view of this we always try to minimize any trace of underachievement and/or overachievement which may sometimes occur. In this work also, considering the assignment of targets; since the interest of the decision maker is to minimize the deviations from the targets as discussed earlier, we now obtain another single model that comprises both the weighted sum and e-constraint terms in which the objective is to minimize the weighted sum of these deviations. The weights are assigned according to the levels of minimization decided by the decision maker. In the linear programming simplex approach, such deviations are called slack variables whereas in MOLP, these slack terms are either positive or negative indicating overachievement or underachievement of the set goals and not only are they real variables but are also the only terms in the objective function which is to be minimized. Hence the resultant program becomes

$$\text{Min } z = W^T d_j$$
$$\text{Subject to } N_j(x) + d_j \geq e_j\ \forall_j \in Ji\ ;$$
$$Ax \leq b;\ x \geq 0;\ d_j \geq 0.$$

Where W = the weight assigned by the decision maker; d_j which is the deviational variable can be d_j^+ or d_j^- depending on the objective of the decision maker; d_j^+ = overachievement of the assigned goal;

d_j^- = is the underachievement of the assigned goal

All these approaches to solutions were demonstrated in the analyses and the results discussed in chapter four.

DATA COLLECTION ON BANK'S INVESTMENT : Every investor must trade off return versus risk in deciding how to allocate his or her available funds. The opportunities that promise the greatest profits are almost always the ones that present the most serious risks. Investment banks must be especially careful in balancing return and risk because legal and ethical obligations demand that they avoid undo hazards. Yet their goal as a business enterprise is to maximize profit and also minimize risk.The investment bank in discussion whose name is not to be mentioned for security and integrity reasons has about ₦20 billion as the capital with ₦150 billion in demand deposits and ₦80 billion in time deposits. The chart (table) below provides every detail about the investment.

	Bank investment category (j)	Return rate (%)	Liquid part (%)	Required capital (%)	Risk asset (%)
1	Cash	0.0	100.0	0.0	No
2	Short term	4.0	99.5	0.5	No
3	Government:2 years	4.5	96.0	4.0	No
4	Government:4 years	5.5	90.0	5.0	No
5	Government:6 years	7.0	85.0	7.5	No
6	Installment loans	10.5	0.0	10.0	Yes
7	Mortgage loans	8.5	0.0	10.0	Yes
8	Commercial loans	9.2	0.0	10.0	Yes

The first goal of any business enterprise is to maximize profit. Hence, using the rates of return from the above table, the objective function becomes

$$\text{Max } P = 0.040X_2 + 0.045X_3 + 0.055X_4 + 0.070X_5 + 0.105X_6 + 0.085X_7 + 0.092X_8 \qquad \text{[profit]}$$

In order to quantify investment risk, we employ the two commonly used ratio measures. The first is the capital adequacy ratio which is expressed as the ratio of the required capital to the actual capital. A low value indicates minimum risk and vise versa. The second objective function becomes

$$\text{Min } C = \tfrac{1}{20}\,(0.005X_2 + 0.040X_3 + 0.050X_4 + 0.075X_5 + 0.100X_6 + 0.100X_7 + 0.100X_8)$$

[capital-adequacy]

Another measure of risk focuses on illiquid risk assets which is measured by the ratio of risk asset to the actual capital and a low ratio indicates a financially secure institution. Hence the third objective function becomes

$$\text{Min } R = \tfrac{1}{20}\,(X_6 + X_7 + X_8) \qquad \text{[Risk-asset]}$$

The following information from the bank presents the relevant constraints.

(1) Investment must sum to the available capital and deposit funds.
(2) Cash reserves must be at least 14% of demand deposit plus 4% of time deposits.
(3) The portion of investments considered liquid should be at least 47% of demand deposits plus 36% of time deposits.
(4) At least 5% of funds should be invested in each of the eight categories, for diversity.
(5) At least 30% funds should be invested in commercial loans, to maintain the bank's community status.

ANALYSIS OF DATA : Combining the three objective functions above with these five systems of constraints, the multiple objective linear programming model of this Bank's investment becomes

$$\text{Max } P = 0.040X_2 + 0.045X_3 + 0.055X_4 + 0.070X_5 + 0.105X_6 + 0.085X_7 + 0.092X_8 \quad \text{[Profit]}$$

$$\text{Min } C = \tfrac{1}{20}\,(0.005X_2 + 0.040X_3 + 0.050X_4 + 0.075X_5 + 0.100X_6 + 0.100X_7 + 0.100X_8)$$

[Capital-adequacy]

$$\text{Min } R = \tfrac{1}{20}\,(X_6 + X_7 + X_8) \qquad \text{[Risk-asset]}$$

Subject to:

$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 = 250$ [Invest all]

$X_1 \geq 24.2$ [Cash reserve]

$1.00X_1 + 0.995X_2 + 0.960X_3 + 0.900X_4 + 0.850X_5 \geq 99.3$ [liquidity]

$X_j \geq 12.5$ for all $j = 1, 2, \ldots, 8$ [diversification]

$X_8 \geq 75$ [commercial]

Here solutions are evaluated on three criteria namely profit, capital-adequacy ratio and risk-asset ratio.

The above problem was solved using the weighted sum approach with a weighting vector $\frac{1}{3}$.

With the set goals for the three criteria as follows

Profit ≥ 18.5

Capital-adequacy ratio ≤ 0.8

Risk-asset ratio ≤ 7.0,

the three objective functions can then be stated in the following manner

Goal 1 $0.040X_2 + 0.045X_3 + 0.055X_4 + 0.070X_5 + 0.105X_6 + 0.085X_7 + 0.092X_8 \geq 18.5$ [profit]

Goal 2 $\tfrac{1}{20}\,(0.005X_2 + 0.040X_3 + 0.050X_4 + 0.075X_5 + 0.100X_6 + 0.100X_7 + 0.100X_8) \leq 0.8$

[capital-adequacy]

Goal 3 $\tfrac{1}{20}\,(X_6 + X_7 + X_8) \leq 7.0$ [risk-asset]

Subject to the constraints as used above. The problem was solved using the e-constraint approach

Also, considering the Bank's investment problem, each of the given goals can either be overachieved or underachieved designated by positive deviational variables and negative deviational variables respectively.

For goal 1, our interest is to minimize underachievement since overachievement increases the profit.

For goals 2 & 3, our interest is to minimize overachievement of risk.

Hence we define the following deviational variables of interest

d_1^- = amount with which profit falls short of its goal

d_2^+ = amount with which capital-adequacy ratio exceeds its goal.

d_3^+ = amount with which risk-asset ratio exceeds its goal.

The objective function which now comprises these deviational variables in a minimization sense is written as

Min Z = $d_1^- + d_2^+ + d_3^+$

Subject to the above given constraints with goal constraints inclusive.

In the analyses, we consider both equal and unequal goal weights.

Using equal goal weights $w_1 = w_2 = w_3 = 1$, the Bank's investment problem has the following linear goal programming model.

Min Z = $d_1^- + d_2^+ + d_3^+$

Subject to

$0.040X_2 + 0.045X_3 + 0.055X_4 + 0.070X_5 + 0.105X_6 + 0.085X_7 + 0.092X_8 + d_1^- \geq 18.5$ [profit]

$0.00025X_2 + 0.002X_3 + 0.0025X_4 + 0.00375X_5 + 0.005X_6 + 0.005X_7 + 0.005X_8 - d_2^+ \leq 0.8$ [capital-adequacy]

$0.05X_6 + 0.05X_7 + 0.05X_8 - d_3^+ \leq 7.0$ [risk-asset]

$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 = 250$ [invest all]

$X_1 \geq 24.2$ [cash reserve]

$1.00X_1 + 0.995X_2 + 0.960X_3 + 0.900X_4 + 0.850X_5 \geq 99.3$ [liquidity]

$X_j \geq 12.5$ for all $j = 1, \ldots, 8$ [diversification]

$X_8 \geq 75$ [commercial]

$X_1, \ldots, X_8, d_1, \ldots, d_3 \geq 0$

Using unequal weights of say

$w_1 = 7, w_2 = 3, w_3 = 1$, we now have

Min Z = $7d_1^- + 3d_2^+ + d_3^+$

Subject to the same constraints as above.

The problems above were solved using the merged approach.

We used two phase method in solving the above problems because of the presence of both equality and inequality signs.

Tora software was also used in solving the problem

FINDINGS : This work comprises analyses on each of the methods of solutions discussed above and the dual results are discussed below as follows:

For the weighted sum approach, even though it converges faster than the e-constraint approach (that is, it has a fewer number of iterations than the e-constraint approach), it does not achieve the targets of the decision maker in the sense that goals are underachieved and/or overachieved; hence the values (-4.458, 5.418, 5) instead of (18.5, 0.8, 7), where -4.458 and 5.418 are non-suported solutions and the 3rd value, 5 is a supported solution. All these values are obtained by substituting the values of the decision variable of the weighted sum optimization in the given objective functions (Klamroth et al, 2003).

For the e-constraint approach; it involves more number of iterations than the other two approaches. It is also more complex to use in the sense that each of the objective functions has to be used individually. Furthermore, it still does not achieve the targets of the decision maker in the sense that it has the underachievement of the profit goal and overachievement of the investment risks; hence the values (17.324, 0.928, 8.01) instead of (18.5, 0.8, 7). All these values are in billions of Naira.

However, for the merged approach in which the objective is to minimize the deviational variables of interest; not only that it has a fewer number of iterations; all the deviational variables $\left(d_1^-, \quad d_2^+, \quad d_3^+\right)$ denoted by (x_9, x_{10} and x_{11}) are minimized to zero showing that all the investment targets are met.

III. CONCLUSION

Applying duality theory to multi objective linear programming is an interesting concept in mathematical programming. Duality generally arises from the fact that most times, it is more difficult to solve a linear programming problem in its original (primal) form and as a result, the dual aspect is generated from the primal and solved.

In this work, to generate solutions to the MOLP problem, we used scalarization methods (approaches) which involve formulating an MOP-related SOP which is a function of the objective functions of the MOP. The three secularization approaches used include: the weighted sum approach, e-constraint approach and the merged weighted sum/e-constraints approach. Each approach was used to solve the same problem with the sole aim of finding which approach handles MOLP problems better and why. Hence a real life example was used to demonstrate the discussion above with theS findings as seen in the discussion of the analyses. Based on these findings, we conclude that multi objective linear programming problems are better and easier solved using duality method applied to the merged approach in which the objective function comprises the deviational variables from the originally given objectives of the decision maker since in most cases it involves fewer or at worst the same number of iterations as in primal problem and all the deviational variables most times are minimized to zero (which is better than either overachievement or underachievement as the case may be) showing that the targets of the decision maker are all met.

IV. RECOMMENDATIONS

The researcher has been able to carry out some analyses using real life collected data and would like to give the following recommendations.

[1] The researcher recommends that the method of solution involving the weighted sum and e-constraint terms is the best for solving MOLP problems since it minimizes the deviational variables to zero or at worst close to zero and also has a fewer number of iterations.

[2] The researcher also recommends that further work should be done on the weighted sum approach so as to know why sometimes it does not provide an optimal solution to least one of the objective functions of an MOLP problem.

[3] Application of duality theory to multi objective linear programming should also be adapted to problems related to everyday life.

[4] Application of duality theory should also be extended to some other areas of soperations research such as convex programming, games theory etc.

[5] Finally, the researcher recommends that the investment bank should increase their rates of return so that they will be able to attain the target set for profit and avoid underachieving it.

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