The International Journal Of Engineering And Science (IJES)
||Volume||2 ||Issue|| 8 ||Pages|| 113-120||2013||
ISSN(e): 2319 – 1813 ISSN(p): 2319 – 1805

Approximate Elastic Model for Determination of Critical Loads and Effective Lengths for Simple Sway Frames

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ABSTRACT
This paper is aimed at providing a simple model for determination of critical loads, for simple sway frames. It is a mathematical model, which is derived systematically from a simplified elastic linear analysis, combined with moment equilibrium at the base of frame at the onset of a mechanism. The deflection due to direct axial compression and the deflection due to shear deformation are neglected to obtain the overall stiffness. Results obtained from the proposed model are striking.

KEYWORDS *- Critical Loads, Effective Length, Linear Elastic Analysis, Sway Frames.*

Date of Submission: 6 , August ,2013 Date of Acceptance:20,August ,2013

I. INTRODUCTION

The determination of the critical loads for sway frames is associated with complicated mathematics. Thus, engineers have tended to develop simplified methods and models for the calculation of critical loads. Indeed, the various design manuals are replete with approximate formulae for determination of critical loads. The investigation of approximate or simplified models for critical loads has received revived attention in the works of Stevens[1], Horne[2], Anderson[3], Wood[4], Goldberg[5], Orumu[6], and Hoenderkamp[7], among several others. The basic approach in these solutions is common. It consists in the assumption of a mechanism at the feet of the frame (Fig.1).

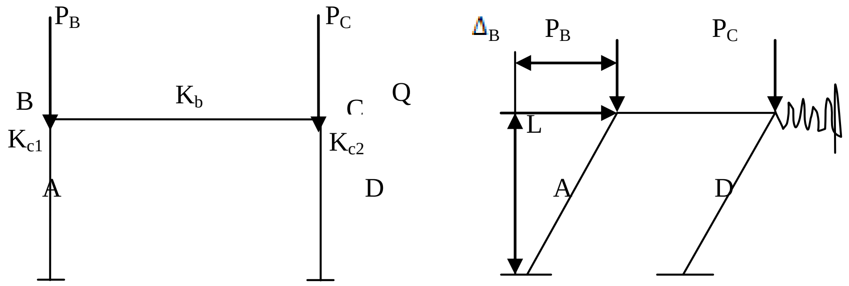


Fig. 1(a) Sway Frame before Critical State Fig. 1 (b) Sway Frame at Critical State

Where n = $\frac{K_b}{K_c}$ j = $\frac{K_{c1}}{K_{c2}}$

The restoring force from sides sway is then determined from the equation of limiting equilibrium about the feet.

$QL - P\Delta_B = KL\Delta_B$

Therefore,

$$\Delta_B = \frac{QL}{(KL - P)}$$

At critical stability condition (instability), P reaches its value Pcr as displacement Δ_B becomes infinitely large. In order words, the denominator, in the limit, becomes zero.

Thus:

$KL - Pcr = 0$
$Pcr = KL$ 1

The major difference in the approaches of various researchers consists in the determination of the frame structural stiffness k. In this regard, Stevens[3] and Horne[8] have applied the rigorous linear elastic analysis, which again involves computer application, repeatable for each solution. On the other hand, Bolton[9] has determined the critical load for frames working from the tabulated no-shear stability functions based on the works of Livesley and Chandler[10]. However, for reasons of their complexity, simple formulae that can be applied to any given frame could not be arrived at.

The author of this paper has derived the stiffness K from a simplified linear elastic analysis akin to the approaches by Stevens and Horne. However, the proposed simplified method enables the stiffness K to be obtained algebraically as a function of the relative stiffness of columns and beams.

II. THE PROPOSED APPROXIMATE ELASTIC MODEL

The adopted approach is to determine the sway Δ_B due to a unit horizontal load Q. The stiffness K is the reciprocal of Δ_B for the unit horizontal load. For simplification, the effect of shear deformation and direct axial compression are neglected. These deflections constitute on the average, about 10% of the total deflections[11]. With this assumption, the element slope deflection equation reduces to:

$M_{AB} = k[4\theta_A+2\theta_B]$ (2)
$M_{BA} = k[2\theta_A+\theta_B]$ (3)

The equilibrium of the frame in structure co-ordinates can be written as:

$\{P\} = \{K_s\}\ \{\delta\}$ (4)

Where

$\{K_s\}$ = structure stiffness matrix; $\{P\}$ = load vector and $\{\delta\}$ = displacement vector.

For a unit lateral load (Q = 1) applied at joint B (Fig. 1b) the proposed simplifies elastic linear analysis to be described subsequently yields

$$\Delta_B = \frac{L^2}{6K_1\ddot{Y}}$$

Where

L = height of frame, K_1 = stiffness of column 1 and $\ddot{Y}$ is a dimensionless parameter which depends on relative beam to column stiffness ratios in the frame.

2.1 The Algebraic Approach

2.1.1 Frame with Rigid Supports

Consider the frame with relative stiffness n and j as defined in Fig. 1 Let d_m and P_m represent global (master) freedoms and corresponding forces while d_s and P_s represent local member (slave) freedoms and corresponding forces. If member forces and freedoms are related through member stiffness K_s as follows:

$P_s = K_sd_s$

The member freedoms are related to the master freedoms for compatibility through the transformation matrix T as follows 3

$D_s = Td_m$

Then using the principle of contragradient transformation.

$P_m = T^TK_sTd_m$

Thus, the structure stiffness becomes

$K = T_TK_sT$

The moment equilibrium equation (4) can be solved using the following procedure

Step 1: Find moments in elements axes: $P_s = K_sd_s$

$$\begin{bmatrix} M_{ab} \\ M_{ba} \\ M_{bc} \\ M_{cb} \\ M_{cd} \\ M_{dc} \end{bmatrix} = K_1 \begin{pmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4n & 2n & 0 & 0 \\ 0 & 0 & 2n & 4n & 0 & 0 \\ 0 & 0 & 0 & 0 & 4j & 2j \\ 0 & 0 & 0 & 0 & 2j & 4j \end{pmatrix} \begin{bmatrix} \theta'_{ab} \\ \theta'_{ba} \\ \theta'_{bc} \\ \theta'_{cb} \\ \theta'_{cd} \\ \theta'_{dc} \end{bmatrix}$$

Step 2: Determine the global degree of freedom DOF
$d_m = [\Delta_B, \theta_B, \theta_C]^T$

Step 3: Set up the system compatibility equation:
$d_m = Td_m$

$$\begin{bmatrix} M_{ab} \\ M_{ba} \\ M_{bc} \\ M_{cb} \\ M_{cd} \\ M_{dc} \end{bmatrix} = K_1 \begin{pmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4n & 2n & 0 & 0 \\ 0 & 0 & 2n & 4n & 0 & 0 \\ 0 & 0 & 0 & 0 & 4j & 2j \\ 0 & 0 & 0 & 0 & 2j & 4j \end{pmatrix} \begin{bmatrix} \theta'_{ab} \\ \theta'_{ba} \\ \theta'_{bc} \\ \theta'_{cb} \\ \theta'_{cd} \\ \theta'_{dc} \end{bmatrix}$$

The global stiffness matrix ($K = T^T K_S T$) after some matrix manipulation, becomes

$$K = \begin{pmatrix} 6(1+j)/L^2 & -3/L & -3j/L \\ 2k_1 -3/L & 2(1+n) & n \\ -3j/L & n & 2(n+j) \end{pmatrix}$$

Where n and j are relative stiffness defined at Fig.1b.

Step 4: Establish the overall equilibrium matrix for the frame using $(1, 0, 0)^T$ as load vector.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2k_1 \begin{pmatrix} 6(1+j)/L^2 & -3/L & -3j/L \\ -3/L & 2(1+n) & n \\ -3j/L & n & 2(n+j) \end{pmatrix} \begin{bmatrix} \Delta_B \\ \theta_B \\ \theta_C \end{bmatrix}$$

Step 5: Determine Δ_B by any appropriate method of solution of simultaneous equations. Using method of the elimination, Δ_B has the expression.

$$\frac{1}{\Delta_B} = \frac{6K_1}{L^2}\left[2(1+j) - \frac{1.5j^2}{(n+j)} - \frac{1.5j\,(n+2j)[2(n+j)-jn]}{(n+j)\,[4(1+n)(n+j)-n^2]}\right]$$

$\Delta_B = \frac{L^2}{6K1\ddot{\Upsilon}}$ 5

Where $\ddot{\Upsilon}$ denotes the expression in expression in figure bracket Global frame stiffness equals.

$K = \frac{6K1}{L^2}\ddot{\Upsilon}$ 6

back substituting eqn. 1 the critical load is determined from

$Pcr = \frac{6K1}{L}\ddot{\Upsilon}$

$Pcr = \frac{6PE}{\pi^2}\ddot{\Upsilon}$ 7

Where n^2 P_E = Cripping load of the column

Now define a load factor λcrs, such that

$P_{cr} = \lambda crs\,(P_B + P_C)$

$P_{cr} = P_B\,(1 + \acute{\alpha})\lambda crs$

Where $\acute{\alpha}$ represents the ratio P_C/P_B

Then from eqn. 7

$\lambda crs = \frac{6\ddot{\Upsilon}\,PE}{PB\,(1+\acute{\alpha})\pi^2}$ 8

2.1.2 Frames with Pinned Supports

If sway frame is pinned at the base, the simplified elastic linear analysis yields the following expression for $\ddot{\Upsilon}$

$\ddot{\Upsilon} = \frac{0.5}{(4n+3j)}\left[4n + 3j + 4nj - \frac{3(4n+3j-2nj)^2}{(3+4n)(4n+3j)-4n^2}\right]$ 9

For symmetrical frame

If j = 1

$\ddot{\Upsilon} = \frac{0.5}{(4n+3)}\left[8n + 3 - \frac{3(2n+3)^2}{(3+4n)(4n+3)-4n^2}\right]$ 10

λcrs is determined from equation 8 by substitution of relevant expression for $\ddot{\Upsilon}$

2.2 Determination of Effective Length

The effective length of the entire frame can be obtained from the known expression.

$L_{eff} = L\sqrt{\frac{PE}{Pcr}}$

Where P_{cr} is the critical load obtained.

2.3 Modification of model

2.3.1 Portal frame with fixed base

The value of $\ddot{\Upsilon}$ from the application for frames rigidly fixed at base varies from 1 to 4 for KB/KC ratio of 0 to ∞ respectively. The expected value of λcr for this range is 0.25 to 1. This is equivalent to the range $\frac{\ddot{\Upsilon}}{\ddot{\Upsilon}_\infty}$ to $\frac{\ddot{\Upsilon}_\infty}{\ddot{\Upsilon}_\infty}$ which means the critical load factor earlier obtained for every KB/KC needs to be multiplied by the ratio of 0.25 and further multiplied by the amplification factor f_{amp} which was derived based on the maximum deflection for KB/KC=0 which is the case of a cantilever.

$f_{amp} = 1 + 0.083333\,\mathrm{Sin}\,\frac{\pi(\ddot{\Upsilon}-1)}{3}$ 11

The modified model becomes

$\lambda crs = \frac{6\ddot{\Upsilon}\,PE}{PB\,(1+\acute{\alpha})\pi^2} x1.214876\, f_{amp}$

$\lambda crs = \frac{0.739\ddot{\Upsilon}\,PE}{PB\,(1+\acute{\alpha})} f_{amp}$ 12

2.3.2 Portal frame with pinned base

The value of Ÿ from the application for frames pinned at base varies from 0 to 1 for KB/KC ratio of 0 to ∞ respectively. The expected value of λcr for this range is 0 to 0.25, which means the critical load factor earlier obtained for every KB/KC needs to be multiplied by the ratio of 0.25 and further multiplied by the amplification factor f_{amp} which was derived based on the maximum deflection for KB/KC=0 which is the case of a cantilever.

$f_{amp} = 1 + 0.083333 \, \mathrm{Sin}\pi\ddot{Y}$ 13

The modified model becomes

$$\lambda cr_s = \frac{6\ddot{Y}\, PE}{PB\ (1+\acute{\alpha})\pi^2} x1.214876\ f_{amp}$$

$$\lambda cr_s = \frac{0.739\ddot{Y}\, PE}{PB\ (1+\acute{\alpha})} f_{amp} \qquad 14$$

III. APPLICATION 1

Find the critical load factor and effective length for various values of n (K_B/K_C) for the frame shown in fig. 1a, where j = 1, if the frame is fixed at base and if the frame is pinned at base.

Solution:

Substituting various value of n for j=1, the results are shown in table 3 and table 4 below will be obtained

Table 3: Values of elastic critical load factor (λcr) for Sway Frame with fixed feet.

n	Ÿ	λcr	λcr modifed	λcr Horne	%diff	effective length
0	1	0.303719	0.25	0.25	0	2
0.1	1.391	0.422565	0.35937394	0.36	-0.174	1.668117
0.25	1.818	0.552216	0.48317233			1.43863
0.5	2.286	0.694215	0.61785371	0.61	0.129	1.272205
0.75	2.588	0.786096	0.70075037			1.194589
0.8	*2.636*	*0.800714*	*0.7134561*	*0.7*	*1.922*	*1.183904*
0.85	2.681	0.814367	0.72518652			1.174289
0.9	2.723	0.827149	0.73604304			1.165596
0.95	2.763	0.839141	0.74611437			1.157703
1	2.8	0.850413	0.7554783	0.75	0.730	1.150506
1.5	3.077	0.934520	0.82198614			1.102980
2	3.25	0.987087	0.86037702			1.078091
4	3.571	1.084711	0.92514016	0.94	-1.581	1.039672
5	3.647	1.107681	0.93921199			1.031854
8	3.769	1.144787	0.96110011	0.97	-0.918	1.020036
9	3.793	1.152038	0.96526347			1.017834
10	3.813	1.157929	0.96862046	0.98	-1.161	1.016069
∞	4	1.214815	1	1	0	1

Table 4: Values of elastic critical load factor (λcr) for Sway Frame with pinned feet

n	Ÿ	λcr	λcr modified	λcr Horne	% diff	effective length
0	0	0	0	0	0	∞
0.1	0.67	0.0506	0.043402	0.04	8.5	4.80000
0.2	0.286	0.0867	0.076082	0.075	1.4	3.62541
0.25	0.5	0.1518	0.13541	0		2.71746
0.5	0.6	0.1822	0.161888	0		2.48537
0.75	0.615	0.1869	0.165833	0		2.45563
0.8	0.63	0.1912	0.169452	0.167	1.468	2.42927
0.85	0.643	0.1952	0.172780	0		2.40575
0.9	0.666	0.2024	0.178695	0		2.36561
1	0.75	0.2277	0.198548	0.184	7.907	2.24423
1.5	0.8	0.2429	0.209796	0		2.18324
2	0.889	0.2699	0.228556	0		2.09172
4	0.909	0.2761	0.232608	0.233	0.168	2.07342
5	0.941	0.2858	0.238897	0		2.04594
8	0.947	0.2877	0.240090	0.242	0.788	2.04085
9	0.952	0.2892	0.241052	0		2.03678
10	0.999	0.3035	0.249907		2.421	2.00036
∞	1	0.3037	0.25	0.25	0	2

3.1 Application 2

Find the critical load factor and effective length for various values of n (K_B/K_C) for the frame shown in fig. 2, for unsymmetrical portal frame i.e j = 0 to ∞, if the frame is fixed at base.

Solution:

Substituting various value of n for j=n, the results are shown in table 5 and table 6 below will be obtained

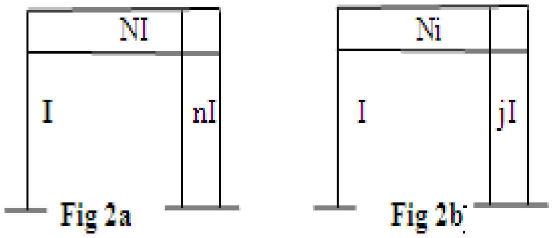


Fig 2a Fig 2b

Table 5: Values of elastic critical load factor (λcr) for Sway Frame with fixed feet. Unsymetrical j=n

n & j	Ÿ	λcr	λcr modifed	effective length	λcr Horne	%diff
0	2	0.6074	0.536084	1.36578		
0.1	2.0241	0.6147	0.543076	1.35696		
0.2	2.0680	0.6281	0.555774	1.34137		
0.25	2.0961	0.6366	0.563861	1.33172		
0.5	2.2826	0.6932	0.61698	1.27310		
0.75	2.5235	0.7664	0.683455	1.20960		
1	2.8	0.8504	0.755478	1.15050	0.75	-0.730
2	4.0909	1.2424	1.01462	0.99276		
4	7	2.1260	1.75	0.75592		
5	8.51	2.58	2.305219	0.65863		
10	16.2	4.93	3.97677	0.50145	4	-0.580
10000	1571.	4773	3805.86	0.01621		

Table 6: Values of elastic critical load factor (λcr) for Sway Frame with fixed feet. Unsymetrical j≠n

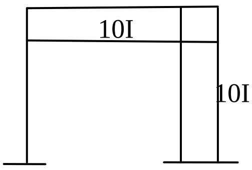
n	j	Ÿ	λcr	λcr modifed	λcr Horne	effective length
0	1	1	0.3037	0.25	0.25	2
0.1	1	1.3913	0.4225	0.35937	0.36	1.6681
0.1	2.5	0.2086	0.0633	0.04896		4.5193
0.1	5	-1.776	-0.539	-0.43545		-
0.1	10	-5.752	-1.747	-1.35302		-
0.25	1	1.8182	0.5522	0.48317		1.4386
0.25	2.5	1.0411	0.3162	0.261207		1.95663
0.25	5	-0.315	-0.095	-0.07232		-
0.25	10	-3.055	-0.928	-0.82059		-
0.5	1	2.2857	0.6942	0.617854	0.61	1.27221
0.5	2.5	2.0352	0.6181	0.546279		1.35298
0.5	5	1.4656	0.4451	0.380718		1.62068
0.5	10	0.2470	0.0750	0.058103		4.14859
1	1	2.8	0.8504	0.755478	0.75	1.15051
1	2.5	3.25	0.9870	0.860377		1.07809
1	5	3.7021	1.1244	0.949202		1.02641
1	10	4.4138	1.3406	1.064838		0.96908
2	1	3.25	0.9871	0.860377		1.07809
2	2.5	4.45	1.352	1.070411		0.96655
2	5	6	1.8223	1.391747		0.84765
2	10	8.7143	2.6467	2.355567		0.65155
5	1	3.6471	1.1077	0.939212	0.94	1.03185
5	2.5	5.6694	1.7219	1.30108		0.87669
5	5	8.5113	2.5851	2.305219		0.65863
5	10	13.493	4.0979	3.511775		0.53362
10	1	3.8125	1.1579	0.96862	0.98	1.01607
10	2.5	6.25	1.8982	1.470429		0.82467
10	5	9.8571	2.9938	2.494893		0.63310
10	10	16.231	4.9296	3.97677	4.3	7.517

IV. DISCUSSION

The results from the worked examples demonstrate the simplicity and accuracy of the proposed mathematical models for simple sway frames. The solutions from Horne and Merchant (2) using the rigorous matrix computer analysis using stability functions are tabulated and used as accurate baseline values for comparison with the critical load and modified critical load factors. The effective lengths for each case are also presented. Tables 3,4,5 and 6 show that the results are very reliable when compared. A rather controversial problem is discussed below For a particular case where the beam is 10I, a stocky column is 10I and a slender column of the sway frame is I, four buckling loads are predicted by the model viz.

From table 6 for

n=1, j=10 $\lambda cr_{\text{modified}}$=1.065
n=10, j=0.10 $\lambda cr_{\text{modified}}$=0.59
n=1, j=0.10 $\lambda cr_{\text{modified}}$=0.57
n=10, j=10 $\lambda cr_{\text{modified}}$=3.98



The two of these critical load factors to be given attention to are the once with integers and not fraction for j i.e
n=1, j=10 $\lambda cr_{\text{modified}}$=1.065
n=10, j=10 $\lambda cr_{\text{modified}}$=3.98

Wood[10] had obtained a value of λcr =4.3 and several discussants of his paper had argued that this value cannot be possibly the critical load and claimed that the computer program may have skipped the fundamental critical load which according to the discussants should be around the critical load factor of the slender column which for the KB/KC ratio of 10 is in the region of λcr =0.98 or slightly above because of the support it will receive from the stocky column. The results obtained in the method herein discussed have validated their arguments. The critical loads Pcr =1.065PE and Pcr =3.98PE appear to be the first and second mode buckling loads of the special frame. The frame will have failed before hitting Pcr =3.98PE speaking from Engineering point of view. Therefore the critical load is Pcr =1.065PE.

In deriving the analytical models, deformations due to axial and shear stress resultants have been ignored. As practice has shown, these components contribute for less than 10 percent of the total stresses in the systems stability response. This, perhaps, also explains the minor disparity between the results from the proposed models and those obtained from the rigorous analytical efforts of Horne and Merchant.

CONCLUSION

The proposed analytical models offer a simplified hand of solution to sway stability problems of building frames. Its success is justified by close reproduction of existing solutions of rigorous methods. Its application will have tremendous impact on reducing cost of computing and increasing scope of solvable problems on sway stability of frames, while opening a greater scope for studies in this rather complex area of structural analysis and design.

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